This 45 min. test can be answered with the aid of any texts, notes and calculating instruments. Experiment freely; answers may be worked out as audaciously as you like on scratch paper but must then be reconsidered before being entered on this sheet in the spaces provided. Each correct proof or algorithm earns two points, or one point if almost correct; other correct answers earn a point each; incorrect answers lose a point; space left blank loses nothing, so **DON'T JUST GUESS.** Finally hand in all your scratch paper, and hand this sheet to your T. A. to be scored and later returned to you. Your score *will* affect your final grade. Don't discuss this test until tomorrow.

1. " $\sum_i x_i^2 = \sum_i y_i^2$ and $\sum_i \sum_j (x_i \cdot y_j - x_j \cdot y_i)^2 \ge 0$. The inequality here becomes equality just when every $x_i = y_i$ or $x_i = -y_i$." This last sentence is potentially ambiguous. Give an example to illustrate what it could mean but shouldn't: __ $x_1 = y_1 = x_2 = -y_2 = 1$ __
Then rewrite the sentence to say unabiguously what it should mean:
__" The inequality here becomes equality just when either every $x_i = y_i$ or every $x_i = -y_i$."__

2. Each of 77 guests at a Christmas party brings a small gift which is put into a big basket; after it is shaken up, the gifts are redistributed to the guests at random and independently, one gift per guest. What is the expected number of guests who will get back their own gift? __ 1 __

3. A die is so loaded that 2, 4 and 6 are each equally likely but twice as likely to appear as each of 1, 3 and 5 .
(a) What is the probability of rolling a 2 ? __ 2/9 __
(b) What is the probability of rolling an even number? __ 6/9 = 2/3 __
(c) What is the probability of rolling 3 even numbers in 6 rolls? __ ${}^6C_3 \cdot (2/3)^3 \cdot (1/3)^3 = 160/729$
Let X be the count of odd numbers rolled and Y the count of even numbers rolled in 6 rolls.
(d) Are X and Y independent? __ NO because X+Y = 6 . __
(e) What is the Expected value of Y ?__ $6 \cdot 2/3 = 4$ __ What is the Variance of Y ? __ 4/3 __

4. A diagonal of a polygon is a line segment joining two non-adjacent vertices.
If integer $n \ge 3$, how many such diagonals does a regular n-sided polygon have? _ $n \cdot (n-3)/2$ _

5. How few people barely suffice to ensure that at least two of them were born on the same day of the week and in the same month though perhaps in different years? __ 85 __

6. How many *different* character strings can be made from all the letters in the word " ANTIGRAFFITI " ? __ $12!/(2! \cdot 2! \cdot 3! \cdot 2!) = 9979200$ __
In how many of these strings do the first and last letters match?__ 9979200/11 = 907200

7. The Expected Values of random variables X and Y over the same sample space satisfy $E(X) \cdot E(Y) = E(X \cdot Y)$. Does this imply X and Y are statistically independent?_ NO _

8. Exhibit a formula for the Expected Value $E((\sum_j y_j^2))$ in terms of the means $\bar{y}_j := E(y_j)$ and (co)variances $c_{ij} := E((y_i - \bar{y}_i) \cdot (y_j - \bar{y}_j))$ of random variables y_j .
__ $E((\sum_j y_j^2)) = \sum_i c_{ii} + \sum_j \bar{y}_j^2$ __

BUT SEE THE CORRECTION BELOW!

9. Three random variables x, y and z are defined as the elements of a triple (x, y, z) that ranges at random with equal probability 1/9 over the nine triples
(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1), (2, 1, 3), (1, 3, 2) .
Choose one of the three random variables x, y or z by circling it.

Is it independent of *each* (separately) of the other variables? __ YES __

Is it independent of *both* (together) of the other variables? __ NO because any two determine the third. __

10. Provide an algorithm by which a computer with memory for less than a hundred numbers can compute the mean and variance of about a million observed numbers x_j each of which will be transmitted to the computer by radio just once, so the computer cannot remember them all.

The simplest algorithm is based upon these formulas: count $n := \sum_j 1$, mean $\bar{x} := \sum_j x_j/n$, and variance $v^2 := \sum_j (x_j-x)^2/n = \sum_j x_j^2/n - \bar{x}^2$. This simplest algorithm goes as follows:

Initialize $j := 0$; $Sx := 0$; $Sx2 := 0$;
While numbers x_j are being received do
{ $j := j+1$; Receive $x := x_j$; $Sx := Sx + x$; $Sx2 := Sx2 + x^2$ } ;
count $n := j$; mean $\bar{x} := Sx/n$; variance $v^2 := Sx2/n - \bar{x}^2$.

Although this algorithm is used widely, when implemented in rounded arithmetic it can produce deceptive results whenever the variance is rather smaller than the squared mean. Simple ways to defend the results against roundoff do exist and are sometimes taught in Math. 128; ask for them.

11. In last week's mail the Math. Dept. received the following original scheme to find out whether a big integer N represented in decimal is divisible by 7, 11 or 13 : " While N is more than three decimal digits wide, let k be the integer represented by the last three decimal digits of N , and let m be obtained from N by deleting from it those last three digits, and replace N by $|m - k|$. The final N at most three digits wide is divisible by 7, 11 or 13 if and only if the original N is divisible by 7, 11 or 13 respectively." Prove that this scheme works.

Proof: Observe that the given $N = 1000\cdot m + k = 1001\cdot m + k-m = 1001\cdot m \pm |k-m|$, and therefore $N \equiv \pm(\text{new } N)$ **mod** 1001 where $(\text{new } N) := |k-m|$. Since $1001 = 7\cdot 11\cdot 13$, we infer that $(\text{new } N) \equiv \pm N$ **mod** 7 , **mod** 11 and **mod** 13 , but (new N)'s decimal representation is not so wide as N's . The last equation persists while the process $N \to (\text{new } N)$ is repeated until the final (new N) is at most three decimal digits wide. Then the final N is divisible by 7, 11 or 13 if and only if the original N is divisible by 7, 11 or 13 respectively, as the scheme intends.

The maximum score on this test is 21 points, but 15 points is an excellent score.

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Choose one of the three random variables x, y or z by circling it.
Is it independent of *both* (together) of the other variables? __NO; they determine it. _
Is it independent of *each* (separately) of the other variables? __ YES __

2. Each of 69 guests at a Christmas party brings a small gift which is put into a big basket; after it is shaken up, the gifts are redistributed to the guests at random and independently, one gift per guest. What is the expected number of guests who will get back their own gift?
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CORRECTION:

Problem 8 was corrupted by a typo "^{2}))" instead of ")^2)" ; it was supposed to read …

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