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*Linear Programming*
We want to maximize an objective function $f(x) = c^{\mathrm{T}}x = c_1x_1 + \cdots c_nx_n$ subject to the constraints $Ax \leq b$ and $x \geq 0$ (the *canonical*, or *normal* form).

- Suppose we have $n$ variables and $m$ constraints. What is the size of the matrix $A$? Vectors $x$, $b$, and $c$?
- Can we have $n < m$? $n = m$? $n > m$? What does each of these mean?
- Feasible region is convex. Why is this important?
- MATLAB's `linprog(f, A, b)`. GNU OCTAVE's `lp(f, A, b)`.
- Another standard way to formluate is the *standard* form: maximize of $c^{\mathrm{T}}x$ under the constraint $Ax = b$ and $x \geq 0$. Show how to convert between the two forms.
- Example. A factory produces two kinds of products $A$ and $B$. It takes 15 hours to make one unit of $A$, and 10 hours to make $B$. Both $A$ and $B$ requires some amount of raw material; suppose that $A$ requires 2 units of raw material, and $B$ takes 8 units. The amount of raw material is limited to 40 units, and the amount of time is limited to 150 hours. However, each unit of $A$ brings in 100 dollars, while $B$ only brings in 70 dollars. How much of $A$ and $B$ should the factory produce in order to maximize its profit? (You may assume that we can make fractional unit of each product).
  1. What are the variables?
  2. What are the constraints?
  3. Write the contraints in the form $Ax \leq b$.
  4. Find the optimal values.
  5. What happens if the profit for product $A$ increases to 105 dollars? 110 dollars?
  6. What happens if we can only make integer amounts of each product?
- More involved example. A post office requires different numbers of full-time employees on different days of the week: on day $i$, $f_i$ full-time employees are required. Union rules state that each full-time employee must work five consecutive days and then receive two days off. (For example, an employee who works Monday to Friday must be off on Saturday and Sunday.) The post office wants to meet its daily requirements using only full-time employees. Formulate an LP to minimize the number of employees that must be hired.
- In general, where does the optimal point fall? Why?
- Is there a situation where no solution exists? What happens geometrically when this happens?
- When is there a unique solution?
- Duality: Minimization of $c^{\mathrm{T}}x$ subject to $Ax = b$ and $x \geq 0$ is equivalent to maximization of $b^{\mathrm{T}}y$ subject to $A^{\mathrm{T}}y \leq c$.

*Simplex Method*
Start at any basic feasible solution $x^{(0)}$ and keep moving to vertices with a better solution.

- Why does the Simplex method work?

- How many hyperplanes do you need to pinpoint a vertex? An edge?
- At any given corner, only $n$ constraints are active in general. Why?
- Moving from one corner to another through an edge interchanges one active constraint. Why?
- We can then proceed as follows. First choose an edge with largest increase per distance (*i.e.*, largest directional derivative). Then move to the vertex on the other side of the chosen edge.
- The above is equivalent to figuring out which of $n$ variables to be non-zero in the standard form.
- Finding the initial solution. Suppose we have a LP problem maximize $c^T x$ subject to $Ax = b$, $x \geq 0$, $b \geq 0$. Consider the modified LP $Ax + z = b$, $x \geq 0$, $z \geq 0$, $b \geq 0$. The minimizing $\sum_i z_i$ subject to the constraints gives a feasible solution to the original problem (if any). Why?

*More Questions*

1. If the feasible region is not convex, what can happen? Give examples where the Simplex algorithm does not find the optimum.

2. How can you handle a constraint of type $|x_1 + x_2| \leq b_1$? What about $|x_1 + x_2| \geq b_1$?

3. Formulate shortest path problem using LP.