Information Processing & Management 20, No. 3, 363–372, 1984
Printed in the U.S.A.

0306-4573/84 $3.00 + .00
Pergamon Press Ltd.

FUZZY PROBABILITIES

LOTFI A. ZADEH†
Computer Science Division, Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720, U.S.A.

Abstract—The conventional approaches to decision analysis are based on the assumption that the probabilities which enter into the assessment of the consequences of a decision are known numbers. In most realistic settings, this assumption is of questionable validity since the data from which the probabilities must be estimated are usually incomplete, imprecise or not totally reliable.

In the approach outlined in this paper, the probabilities are assumed to be fuzzy rather than real numbers. It is shown how such probabilities may be estimated from fuzzy data and a basic relation between joint, conditional and marginal fuzzy probabilities is established. Manipulation of fuzzy probabilities requires, in general, the use of fuzzy arithmetic, and many of the properties of fuzzy probabilities are simple generalizations of the corresponding properties of real-valued probabilities.

1. INTRODUCTION

Computer-assisted decision analysis is likely to play an increasingly important role in applications—such as command and control—in which vast amounts of information in the database exceed the capacity of the unassisted human mind to assess the consequences of various alternatives and choose that action which is optimal with respect to a set of specified criteria.

An issue which complicates the determination of an optimal action is that in most realistic settings the information which is available to the decision maker is imprecise, incomplete or not totally reliable. In the conventional approaches to decision analysis, uncertain information is treated probabilistically, with probabilities assumed to be known in numerical form. In an alternative approach which is described in this paper, a more realistic assumption is made, namely, that probabilities are known imprecisely as fuzzy rather than crisp numbers. Such probabilities—which will be referred to as *fuzzy probabilities*—are exemplified by the perceptions of likelihood which are commonly labeled as *very likely*, *unlikely*, *not very likely*, etc. [29].

The concept of fuzzy probability is distinct from that of second-order probability (i.e. a probability-value which is characterized by its probability distribution) and contains that of interval-valued probability as a special case. As will be seen in the sequel, in our formulation of the concept of fuzzy probability, the uncertainty in probability is characterized by a possibility[26] rather than probability distribution. In our view, the use of possibility rather than probability in this context leads to a more effective way of dealing with uncertain probabilities and provides a basis for a natural generalization of classical probability theory.

2. THE CONCEPT OF FUZZY PROBABILITY

Theories of subjective probability provide ways of eliciting probability judgments but do not have much to say about the processes by which such judgments are formed. To relate this issue to the concept of fuzzy probability, it is instructive to consider an elementary example of a situation in which an imprecise perception of probability is formed and in which the underlying uncertainty is possibilistic rather than probabilistic in nature. Specifically, consider the following question, in which the italicized words have a fuzzy

† Research supported by NSF Grant IST-8320416 and NESC Contract N0039-83-C-0243.

meaning: An urn contains *approximately n* balls of various sizes, of which *several* are *large*. What is the probability that a ball drawn at random is *large*?

If the information-bearing terms in the question, namely, *approximately n*, *several* and *large* were real numbers, the answer to the question would be a numerical probability. But, since these terms are fuzzy rather than real numbers, it should be expected that the desired probability, like the data on which it is based, is a fuzzy number, i.e. a fuzzy probability.

This conclusion may be stated more succinctly as a principle, namely: *Fuzzy information induces fuzzy probabilities.* What is important to note is that this principle is at variance with the traditional Bayesian point of view which implies that probabilities are real numbers regardless of the nature of the underlying data.

To support our principle, we have to be able to demonstrate how fuzzy probabilities may be computed from fuzzy data. To this end, it is necessary, first, to provide a mechanism for counting the number of elements in a fuzzy set, i.e. for determining its cardinality or, more generally, its measure. We need this mechanism to precisiate the meaning of fuzzy descriptions such as *several large balls* and to be able to answer fuzzy questions like, "How many *combat-ready* ships are there in the *vicinity* of the Persian Gulf?"

In what follows, we shall limit our discussion of the cardinality of fuzzy sets to what is needed to enable us to define fuzzy probabilities in more concrete terms than we have done so far. A more detailed discussion of related issues may be found in [25, 27 and 28].

Consider a fuzzy set A which is represented as

$$A = \mu_1/u_1 + \ldots + \mu_n/u_n \tag{1.1}$$

where μ_i, $i = 1, \ldots, n$, is an element of a universe of discourse U, μ_i is the grade of membership of u_i in A, and $+$ denotes the union rather than the arithmetic sum.

Strictly speaking, it is not meaningful to ask for a count of the elements of U which are in A, since some of the elements of U may be in A "to a degree." Nevertheless, it is useful to have one or more extensions of the conventional concept of cardinality which make it meaningful to speak of the "count" of elements of a fuzzy set. One such extension, which was suggested by DELUCA AND TERMINI[3], is the power of A, which is defined as the arithmetic sum of the grades of membership in A of all elements of U. For our purposes, it is preferable to refer to this count as the sigma-count of A and write

$$\Sigma\,\text{Count}\,(A) \stackrel{\Delta}{=} \Sigma_i \mu_i \tag{1.2}$$

with the understanding that, when appropriate, the right-hand member of (1.2) may be rounded to the nearest integer.

Example: Assume that U is comprised of the elements a, b, c, d, e and f. Then

$$\Sigma\,\text{Count}\,(0.8/a + 0.3/b + 0.8/c + 1/d + 0.2/e) = 3.$$

An alternative extension which was suggested in [25] defines the count of A as a fuzzy number. More specifically, let A_α be the α-level-set of A, i.e. the nonfuzzy set defined by [24]

$$A_\alpha \stackrel{\Delta}{=} \{u_i | \mu_A(u_i) \geq \alpha\}\ 0 > \alpha \geq 1,\ u_i \in U,\ i = 1, \ldots, n, \tag{1.3}$$

where $\mu_i \stackrel{\Delta}{=} \mu_A(u_i)$, $i = 1, \ldots, n$, is the grade of membership of u_i in A. Then, as shown in [24], A may be expressed in terms of the A_α by the relation

$$A = \Sigma_\alpha \alpha A_\alpha, \tag{1.4}$$

where Σ stands for the union, and αA_α is a fuzzy set whose membership function is defined by

$$\mu_{\alpha A_\alpha}(u) = \alpha \text{ for } u \in A_\alpha$$
$$= 0 \text{ elsewhere.} \qquad (1.5)$$

For example, if $U = \{a, b, c, d, e, f\}$ and

$$A = 0.8/a + 0.3/b + 0.8/c + 1/d + 0.2/e \qquad (1.6)$$

then

$A_1 = \{d\}$; $A_{0.8} = \{a, c, d\}$; $A_{0.3} = \{a, b, c, d\}$; $A_{0.2} = \{a, b, c, d, e\}$ and (1.4) becomes

$$A = 1/d + 0.8/(a + c + d) + 0.3/(a + b + c + d) + 0.2/(a + b + c + d + e). \qquad (1.7)$$

Now, let Count(A_α) denote the count of elements of the nonfuzzy set A_α. Then, the *FG*Count of A, where F stands for *fuzzy* and G stands for greater than, is defined as the fuzzy number

$$FG\text{Count}(A) \stackrel{\Delta}{=} \Sigma_\alpha \alpha/\text{Count}(A_\alpha) \qquad (1.8)$$

with the understanding that any gap in the Count (A_α) may be filled by a lower count with the same α. For example, for A defined by (1.6), we have

$$FG\text{Count}(A) = 1/1 + 0.8/3 + 0.6/4 + 0.2/5$$
$$= 1/0 + 1/1 + 0.8/2 + 0.8/3 + 0.6/4 + 0.2/5. \qquad (1.9)$$

Let $A\downarrow$ denote A sorted in descending order and let $NA\downarrow$ denote the fuzzy number resulting from replacing the mth element in $A\downarrow$ by μ_m/m and adding the element 1/0. For example, if

$$A = 0.6/a + 0.9/b + 1/c + 0.6/d + 0.2/e \qquad (1.10)$$

then

$$A\downarrow = 1/c + 0.9/b + 0.6/a + 0.6/d + 0.2/e \qquad (1.11)$$

$$NA\downarrow = 1/0 + 1/1 + 0.9/2 + 0.6/3 + 0.6/4 + 0.2/5. \qquad (1.12)$$

In terms of this notation, then, the definition of FGCount (A) stated earlier (1.8) may be expressed more succinctly as

$$FG\text{Count}(A) = NA\downarrow. \qquad (1.13)$$

To illustrate the use of the concepts defined above, we shall show how to arrive at an answer to a question raised earlier, namely, "How many combat-ready ships are there in the vicinity of the Persian Gulf?"

Assume that in the database we have a relation LIST [Name; μCR; μProx] which lists the name of each ship; its degree of combat-readiness, μCR; and its degree of proximity to the Persian Gulf, μProx.

Furthermore, we assume that only those ships are to be considered whose degree of proximity to the Persian Gulf exceeds a specified threshold, say 0.6.

From the relation LIST we can derive another relation LIST[Name; μCR $\wedge$ μProx], in which μCR $\wedge$ μProx denotes the combined degree of combat-readiness and proximity to the Persian Gulf. For example, if the degrees of combat-readiness and proximity of a ship S1 are 0.8 and 0.7, respectively, then the combined degree will be assumed to be the smaller of the two degrees, i.e. 0.7. More generally, in place of $\wedge$ (min) any desired mode of aggregation may be employed to express the combined degree as a function of its constituents.

To be specific, assume that the relation LIST[Name; μCR/μProx] reads

LIST	Name	$\mu\text{CR} \wedge \mu\text{Prox}$
	S1	0.9
	S2	1
	S3	0.6
	S4	0.7
	S5	0.8
	S6	0.7
	S7	0.9
	S8	1
	S9	0.8

Upon using (1.2), the Σ-count of LIST is found to be given by:

$$\Sigma\text{Count (LIST)} = 8.$$

On the other hand, using the definition of *FG*Count, we obtain the fuzzy number

$$FG\text{Count (LIST)} = 1/0 + 1/1 + 1/2 + 0.9/4 + 0.8/5$$
$$0.8/6 + 0.7/7 + 0.7/8 + 0.6/9.$$

A constituent such as 0.8/6 in this number signifies that there are six ships whose combined degree of combat-readiness and proximity to the Persian Gulf is greater than or equal to 0.8.

In addition to providing a basis for answering questions of the form "How many objects are there which satisfy a set of specified fuzzy criteria?", the concept of cardinality serves also as a means of precisiation of descriptions of the form QAO, where Q is a fuzzy quantifier, e.g. *several, many, few*, etc.; A is a fuzzy adjective, e.g. *tall, combat-ready, blue, young*, etc.; and O is the description of an object, e.g. *ball, ship, car, man*, etc. As will be seen in the sequel, the ability to precisiate the meaning of such expressions plays an essential role in the computation of fuzzy probabilities.

As a concrete illustration, consider the description

$$d \overset{\Delta}{=} \text{several large balls.} \tag{1.14}$$

Using test-score semantics[28], the meaning of d may be defined as a test procedure which yields the degree of compatibility of d with a database which consists of a collection $D = \{b_1, \ldots, b_m\}$ of m balls of various sizes. More specifically, let $\mu_{\text{LARGE}}(b_i)$ be the degree to which a ball, b_i, $i = 1, \ldots, m$, is large. Furthermore, let μ_{SEVERAL} denote the membership function of the fuzzy quantifier *several*. Then, on employing the Σ-count of large balls, we have

$$\Sigma\text{Count (LARGE BALL)} = \Sigma_{i=1}^{m}\, \mu_{\text{LARGE}}(b_i) \tag{1.15}$$

and hence the degree to which this count satisfies the constraint induced by the quantifier *several* is given by

$$\tau = \mu_{\text{SEVERAL}}(\Sigma_i^m\, \mu_{\text{LARGE}}(b_i)). \tag{1.16}$$

As shown in [27], the degree of compatibility, τ, may be interpreted as the possibility of the database, D, given the description $d \overset{\Delta}{=}$ *several large balls*.

Alternatively, the compatibility of d with D may be computed by using the *FG*Count.

Thus, let τ_1

$$\tau_1 \triangleq \mu_{\text{LARGE}}(b_1) \wedge \ldots \wedge \mu_{\text{LARGE}}(b_m)$$

represent the degree to which the constraint on the size of the balls is satisfied. Now, the degree to which the constraint on the number of balls is satisfied is given by

$$\tau_2 = \mu_{\text{SEVERAL}}(m) \tag{1.17}$$

and hence the degree to which both constraints are satisfied may be expressed as

$$\begin{aligned}\tau &= \tau_1 \wedge \tau_2 \\ &= \mu_{\text{SEVERAL}}(m) \wedge \mu_{\text{LARGE}}(b_1) \wedge \ldots \wedge \mu_{\text{LARGE}}(b_m),\end{aligned} \tag{1.18}$$

where $\wedge$ denotes the min operator in infix form. This expression for the compatibility of d and D corresponds to what is referred to in [28] as a *compartmentalized* interpretation of the description $d \triangleq$ *several large balls.*

The foregoing analysis provides us with a means of precisiating the meaning of descriptions of the general form QAO, of which $d \triangleq$ *several large balls* is a typical instance. With this means, then, we can address the issue of computing fuzzy probabilities when the underlying data contain fuzzy descriptions.

As a concrete illustration, we shall consider a slightly simplified version of a question that was posed earlier. Specifically:

An urn contains m balls of various sizes, of which several are large. What is the probability that a ball drawn at random is large?

The simplification—which is not essential—is that the number of balls in the urn is assumed to be m rather than *approximately* m. Furthermore, in the representation of the meaning of *several large balls* we shall employ the ΣCount rather than the *FG*Count.

Assume that the urn, U, consists of the balls $b_1, \ldots, b_m$, with $\mu_{\text{LARGE}}(b_i)$, $i = 1, \ldots m$, representing the grade of membership of b_i in the fuzzy set LARGE. Now from (1.16) it follows that the possibility of U given the datum "The urn contains several large balls" is

$$\tau = \mu_{\text{SEVERAL}}(\Sigma_{i=1}^{m} \mu_{\text{LARGE}}(b_i)).$$

On the other hand, if a ball is chosen at random, the probability of the fuzzy event "The chosen ball is large" is given by (see [23])

$$q \triangleq \text{Prob}\{\text{ball is large}\} = \frac{1}{m} \Sigma_{i=1}^{m} \mu_{\text{LARGE}}(b_i). \tag{1.19}$$

Consequently, the possibility that Prob{ball is large} may take a value, say, v, is

$$\text{Poss}\{q = v\} = \mu_{\text{SEVERAL}}(mv) \tag{1.20}$$

or equivalently,

$$\Pi_q = \frac{\text{SEVERAL}}{m}, \tag{1.21}$$

where Π_q denotes the possibility distribution of q [25] and SEVERAL is interpreted as a fuzzy number. For example, if $m = 10$ and

$$\text{SEVERAL} = 0.4/3 + 0.8/4 + 1/5 + 1/6 + 0.6/7 + 0.3/8 \tag{1.22}$$

then

$$\Pi_q = 0.4/0.3 + 0.8/0.4 + 1/0.5 + 1/0.6 + 0.6/0.7 + 0.3/0.8 \qquad (1.23)$$

is the possibility distribution of the fuzzy number which represents the fuzzy probability that a ball drawn at random is large.

As was pointed out earlier, the fuzziness in the probability of drawing a large ball is induced by the fuzziness in our knowledge of the number of large balls in the urn. In this connection, it should be noted that, if instead of being given the ΣCount of large balls we were given the *FG*Count of large balls, the expression for fuzzy probability would become

$$F\text{Prob}\{\text{ball is large}\} = \frac{FG\text{Count(LARGE)}}{m}, \qquad (1.24)$$

where *F*Prob identifies the probability in question as a fuzzy probability which is the ratio of the fuzzy number *FG*Count (LARGE) and the nonfuzzy number m. Stated in this form, the fuzzy probability *F*Prob{ball is large} becomes closely related to the probability distribution function of the random variable which is associated with the membership function of LARGE.

Remark. There is a significant difference between the results expressed by (1.21) and (1.24) that is in need of clarification.

In the case of (1.21), it is tacitly assumed that the probability of drawing a large ball is a real number whose possibility distribution is expressed by (1.21). In the case of (1.24), on the other hand, the probability is assumed to be a fuzzy rather than a real number. To differentiate between these interpretations, the probabilities expressed by (1.21) and (1.24) will be referred to as *disjunctive fuzzy probability* and *conjunctive fuzzy probability*, respectively. We shall rely on the context to indicate whether a fuzzy probability should be interpreted in a disjunctive or conjuctive sense.

To view the computation of fuzzy probability from a broader perspective, let $U = \{u_1, \ldots, u_m\}$ be a finite universe of discourse and let X be a variable which takes the values $u_1, \ldots, u_m$ with a uniform probability $(1/m)$. Now if A is a nonfuzzy subset of U, the probability of the proposition or, equivalently, of the event

$$p \triangleq X \in A \qquad (1.25)$$

is given by

$$\text{Prob}\{X \in A\} = \frac{\text{Count}(A)}{m}. \qquad (1.26)$$

More generally, if A is a fuzzy subset of U then the probability of the fuzzy proposition or, equivalently, of the fuzzy event

$$p \triangleq X \text{ is } A \qquad (1.27)$$

may be expressed in two distinct ways: (a) as a nonfuzzy probability

$$\text{Prob}\{X \text{ is } A\} \triangleq \frac{\Sigma\text{Count}(A)}{m} \qquad (1.28)$$

and (*b*) as a fuzzy probability

$$F\text{Prob}\{X \text{ is } A\} \triangleq \frac{FG\text{Count}(A)}{m} \qquad (1.29)$$

with the understanding that (1.29) is implied by

$$\text{Prob}\{X \in A_\alpha\} = \frac{\text{Count}(A_\alpha)}{m} \qquad (1.30)$$

where A_α is the α-level-set of A.

Furthermore, if A and B are fuzzy subsets of U, then the joint fuzzy probability of the fuzzy events $p \triangleq X$ is A and $q \triangleq X$ is B is given by

$$F\text{Prob}\{X \text{ is } A,\ X \text{ is } B\} = \frac{FG\text{Count}(A \cap B)}{m}. \quad (1.31)$$

Correspondingly, the conditional fuzzy probability of p given q may be defined as the fuzzy number

$$F\text{Prob}\{X \text{ is } A | X \text{ is } B\} = \Sigma_\alpha\, \alpha \bigg/ \frac{\text{Count}(A_\alpha \cap B_\alpha)}{\text{Count}(B_\alpha)} \quad (1.32)$$

where Σ stands for the union rather than the arithmetic sum and Count $(B_\alpha) \neq 0$.

From (1.31) and (1.32), we can deduce the basic identity for fuzzy probabilities:

$$F\text{Prob}\{X \text{ is } A,\ X \text{ is } B\} = F\text{Prob}\{X \text{ is } A\} \otimes F\text{Prob}\{X \text{ is } B\} \quad (1.33)$$

where $\otimes$ is the product of fuzzy numbers[4, 14]. This identity may be viewed as a natural generalization of the familiar relation:

$$\text{Prob}\{X \in A,\ X \in B\} = \text{Prob}\{X \in B\}\ \text{Prob}\{X \in A | X \in B\} \quad (1.34)$$

which holds when A and B are nonfuzzy probabilities of fuzzy events defined via the ΣCount as in (1.28).

In the foregoing discussion, we have assumed that the probability distribution on U is uniform. More generally, if $\text{Prob}(u_i) \triangleq p_i$, $i = 1, \ldots, m$, and $p_1 + \ldots + p_m = 1$, then (1.29) becomes

$$F\text{Prob}\{X \text{ is } A\} = \Sigma_\alpha\, \alpha / \text{Prob}(A_\alpha), \quad (1.35)$$

where

$$\text{Prob}(A_\alpha) \triangleq \text{Prob}\{X \in A_\alpha\}. \quad (1.36)$$

Similarly, if X and Y take values in $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$, respectively, and

$$\text{Prob}(u_i, v_j) \triangleq p_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

then

$$F\text{Prob}\{X \text{ is } A,\ Y \text{ is } B\} = \Sigma_\alpha\, \alpha / \text{Prob}\{X \in A_\alpha,\ Y \in B_\alpha\} \quad (1.37)$$

$$F\text{Prob}\{X \text{ is } A | Y \text{ is } B\} = \Sigma_\alpha\, \alpha / \text{Prob}\{X \in A_\alpha | Y \in B_\alpha\} \quad (1.38)$$

and

$$F\text{Prob}\{X \text{ is } A,\ Y \text{ is } B\} = F\text{Prob}\{Y \text{ is } B\} \otimes F\text{Prob}\{X \text{ is } A | Y \text{ is } B\}, \quad (1.39)$$

which reduces to (1.33) when $X = Y$. By analogy with numerical probabilities, the fuzzy events X is A and Y is B will be said to *independent* if

$$F\text{Prob}\{X \text{ is } A | Y \text{ is } B\} = F\text{Prob}\{X \text{ is } A\}, \quad (1.40)$$

which implies that

$$F\text{Prob}\{X \text{ is } A,\ Y \text{ is } B\} = F\text{Prob}\{X \text{ is } A\} \otimes F\text{Prob}\{Y \text{ is } B\}. \quad (1.41)$$

In addition to the cases discussed above, there is another important way in which fuzzy information induces fuzzy probabilities. More specifically, consider the case where an individual, I, is faced with the decision of whether or not to insure his or her car. To find that choice which maximizes the expected utility, it is necessary to know, among other parameters, the probability that I's car may be stolen. How could this probability be determined?

Just a little reflection makes it clear that the desired probability cannot be deduced from the statistics of car thefts, since I's car—and the way in which it is driven, parked and garaged—is unique. Furthermore, there is no way in which I can obtain the desired probability by experimentation. What we see in this case is a paradigm of a well known paradox in probability theory which raises serious questions with regard to the meaningfulness of the concept of probability in application to unique events.

One way of getting around the difficulty with uniqueness is to relate probability to similarity, with similarity viewed as a fuzzy relation[24]. Thus, suppose that we wish to estimate the probability that an object, a, which is an element of a finite universe of discourse U, belongs to A, a subset of U. To this end, let S be a fuzzy similarity relation which associates with each element $u \in U$ its degree of similarity to a, $\mu_S(u, a)$. Then, $\mu_S(u, a)$ may be regarded as the membership function of a fuzzy set, $S(a)$, of objects which are similar to a.

In terms of the similarity relation S, the probability Prob $\{a \in A\}$ may be defined via the ΣCount or the FGCount of the intersection of $S(a)$ and A. Thus

$$\text{Prob}\{a \in A\} \triangleq \frac{\Sigma\text{Count}\,(S(a) \cap A)}{\text{Count}\,(U)} \tag{1.42}$$

or, alternatively,

$$\text{Prob}\{a \in A\} \triangleq \frac{FG\text{Count}\,(S(a) \cap A)}{\text{Count}\,(U)} \tag{1.43}$$

where $S(a) \cap A$ denotes the intersection of the fuzzy set $S(a)$ with A [23].

If the similarity relation were known precisely, (1.42) would yield a numerical value for the probability of a belonging to A. But, in general, this would not be the case, with the result that the expression for Prob $\{a \in A\}$ would be a fuzzy number. In this sense, then, estimates of probability based on similarity will, in general, be fuzzy rather than real numbers.

To make the point more concretely, assume that the imprecision in S is modeled by treating S as a fuzzy relation of type 2[26], which implies that the degree of similarity of a and u, $\mu_S(a, u)$, is taken to be a fuzzy number. More specifically, assume that $\mu_S(a, u)$ is a fuzzy number ϕ whose membership function μ_ϕ is a π-function [25], that is,

$$\begin{aligned}
\pi(u\ \delta, \gamma) &= o \text{ for } u \leq \gamma - \delta \\
&= 2\left(\frac{u - \gamma + \delta}{\delta}\right)^2 \text{ for } \gamma - \delta \leq \gamma - \frac{\delta}{2} \\
&= 1 - 2\left(\frac{\mu - \gamma}{\delta}\right)^2 \text{ for } \gamma - \frac{\delta}{2} \leq u \leq \gamma \\
&= 1 - 2\left(\frac{\gamma - u}{\delta}\right)^2 \text{ for } \gamma \leq u \leq \gamma + \frac{\delta}{2} \\
&= 2\left(\frac{\gamma + \delta - u}{\delta}\right)^2 \text{ for } \gamma + \frac{\delta}{2} \leq u \leq \gamma + \delta \\
&= 0 \text{ for } u \geq \gamma + \delta
\end{aligned} \tag{1.44}$$

where γ is the peak of ϕ and δ is its bandwidth. Then, as shown in [4], the sum of fuzzy numbers of this form is a number of the same form whose peak and bandwidth are, respectively, the arithmetic sums of the peaks and bandwidths of their summands. In this way, the numerator of (1.42) evaluates to a fuzzy number which upon division by the count of U yields the fuzzy probability of a belonging to A.

Since the principle of maximization of expected utility plays a central role in decision analysis, it is important to be able to evaluate fuzzy expectations of the general form

$$E = (g_1 \otimes p_1) \oplus (g_2 \otimes p_2) \oplus \cdots \oplus (g_n \otimes p_n), \tag{1.45}$$

where $\otimes$ and $\oplus$ represent fuzzy multiplication and addition, respectively; $p_1, \ldots, p_n$ are fuzzy probabilities; and $g_1, \ldots, g_n$ are fuzzy gains (or utilities). Expressions of the form (1.45) can readily be computed by the use of fuzzy arithmetic. This and related issues are discussed in greater detail in [1, 2, 4, 5, 7, 8, 11, 14, 15, 18, 19, 20, 22] and other papers in the literature.

3. CONCLUDING REMARK

The main point which we have attempted to convey in this paper is that in most realistic applications of decision analysis the underlying probabilities are fuzzy rather than real numbers. In general, fuzzy probabilities are induced by fuzzy data and may be determined by (a) employing the concept of cardinality of fuzzy sets, and (b) using fuzzy arithmetic to compute the ratios, products and sums of counts of elements in such sets.

REFERENCES

[1] J. M. Adamo, Fuzzy decision trees, *Fuzzy Sets and Systems* 1980 **4** 207–219.

[2] S. J. Baas and H. Kwakernaak, Rating and ranking of multi-aspect alternatives using fuzzy sets, *Automatica* 1977 **13** 47–58.

[3] A. DeLuca and S. Termini, A definition of non-probabilistic entropy in the setting of fuzzy sets theory, *Information and Control* 1972 **20** 301–312.

[4] D. Dubois and H. Prade, Operations on fuzzy numbers, *Int. J. Syst. Sci.* 1978 **9** 613–626.

[5] D. Dubois and H. Prade, Decision-making under fuzziness. In *Advances in Fuzzy Set Theory and Applications* (Edited by M. M. Gupta, R. K. Ragade and R. R. Yager) North-Holland, Amsterdam, 279–302, 1979.

[6] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York, 1980.

[7] J. Efstathiou and V. Rajkovic, Multiattribute decision-making using a fuzzy heuristic approach, *IEEE Trans. Syst. Man, Cybern*, 1979 **SMC-9** 326–333.

[8] J. Efstathiou and R. M. Tong, Ranking fuzzy sets using linguistic preference relations, *Proc. 10th Int. Symp. Multiple-Valued Logic*, Northwestern Univ., Evanston, IL, 1980.

[9] T. Fine, *Theories of Probability*. Academic Press, New York, 1973.

[10] P. C. Fishburn, *Mathematics of Decision Theory*. Mouton, The Hague, 1973.

[11] R. Jain, Decision-making in the presence of fuzzy variables, *IEEE Trans. on Systems, Man and Cybern.* 1976 **SMC-6,** 698–703.

[12] R. L. Keeney and R. Raiffa, *Decisions with Multiple Objectives: Preferences and Value Trade-offs*. Wiley, New York, 1976.

[13] W. J. M. Kickert, *Fuzzy Theories on Decision Making: A Critical Review*. Martinus Nijhoff, Leiden, Netherlands, 1978.

[14] M. Mizumoto and K. Tanaka, Some properties of fuzzy numbers, in *Advances in Fuzzy Set Theory and Applications* (Edited by M. M. Gupta, R. K. Ragade and R. R. Yager). North-Holland, Amsterdam, 153–164, 1979.

[15] H. T. Nguyen, On fuzziness and linguistic probabilities, *J. Math. Anal. and Appl.* 1977 **61,** 658–671.

[17] R. I. Savage, *Statistics: Uncertainty and Behavior*. Houghton–Mifflin, Boston, 1968.

[18] J. Tanaka, T. Okuda and K. Asai, Fuzzy information and decision in statistical model, In *Advances in Fuzzy Set Theory and Applications* (Edited by M. M. Gupta, R. K. Ragade and R. R. Yager), 300–320. North–Holland, Amsterdam, 1979.

[19] R. M. Tong and P. P. Bonissone, A linguistic approach to decision-making with fuzzy sets, *IEEE Trans. on Systems, Man, and Cybernetics* 1980 **SMC–10** 716–723.

[20] S. R. Watson, J. J. Weiss and M. L. Donell, Fuzzy decision analysis, *IEEE Trans. Syst. Man, Cybern*, 1979 **SMC–9** 1–9.

[21] J. W. WILCOX, A Method for Measuring Decision Assumptions. MIT Press, Cambridge, 1972.
[22] R. R. YAGER, Fuzzy sets, probabilities and decision, *J. Cybernetics*, 1980 **10** 1–18.
[23] L. A. ZADEH, Probability measures of fuzzy events, *J. Math. Anal. Appl.* 1968 **23** 421–427.
[24] L. A. ZADEH, *Similarity relations and fuzzy orderings, Inf. Sci.* 1971 **3** 177–200.
[25] L. A. ZADEH, A Theory of approximate reasoning, *Electronics Research Laboratory Memorandum M* 77/58, University of California, Berkeley, 1977. Also in *Machine Intelligence* 9 (Edited by J. E. Hayes, M. Michie and L. I. Kulich). 149–194, Wiley, New York, 1979.
[26] L. A. ZADEH, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1978 **1** 3–28.
[27] L. A. ZADEH, Possibility theory and soft data analysis, *Electronics Research Laboratory Memorandum M*79/58, University of California, Berkeley, 1979. Also in *Mathematical Frontiers of the Social and Policy Sciences* (Edited by L. Cobb and R. M. Thrall) 69–120, Boulder, Westview Press, 1981.
[28] L. A. ZADEH, Test-score semantics and meaning representation via PRUF, Tech. Note 247, AI Center, SRI International, Menlo Park, 1981. Also in *Empirical Semantics* (Edited by B. B. Rieger). Bochum: Brockmeyer, 281–349, 1982.
[29] L. A. ZADEH, The concept of a linguistic variable and its application to approximate reasoning, Part I, *Inf. Sci.* 1975, **8**, 199–249; Part II, *Inf. Sci.* 1975, **8**, 301–357; Part III, *Inf. Sci.* 1975, **9**, 43–80.