Probability Criterion for the Design of Servomechanisms

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Of the criteria that are used in the design of servomechanisms, the so-called "minimum mean-square" error criterion is used quite widely in cases where the undesirability of the error increases with its magnitude. There is, however, an important class of applications, particularly in the field of ballistics, where a more appropriate criterion can be defined as the probability that the error be less than some prescribed tolerance. The problem of maximization of this probability is shown to reduce to a number of implicit equations whose solution in practice must be carried out by cut and trial. It is further shown that under certain conditions the maximization of probability yields the same values for the design constants as the minimization of mean-square error. These conditions are discussed for the general case as well as for certain specific cases which are important in practice.

I. INTRODUCTION

THE design of servomechanisms is usually based on some generally accepted criterion of performance. A relatively simple and yet useful one which was introduced some years ago[1] is based on minimization of the mean-square error. This criterion is applicable in situations where the undesirability of the error increases with its magnitude. While this is the case in most common applications, there is an important class of problems in ballistics and certain industrial applications where it is desirable that the magnitude of the error, ϵ, be less than a certain critical value, L. Thus, in these problems, all errors larger than L are equally bad, while those smaller than L are equally acceptable. It is apparent that for such cases the mean-square error criterion does not reflect correctly the requirements of the system.

As a measure of performance for this class of problems, the authors have been using a figure of merit, p, which is defined by the following functional

$$p = \text{Prob.}\{|\epsilon| < L\}. \tag{1}$$

The difficulty arising in the application of (1) is that the complexity of the functional usually prohibits purely analytical approaches such as are possible in the case of the mean-square error criterion.[2-4] However, it is shown in this paper that in many problems where the *a priori* requirement of the design is that p be maximum, the relative values of systematic error, random error, and the tolerance L are such that practically the same design is obtained as through application of the mean-square error criterion. In the following sections a criterion for the equivalence of the two design procedures is obtained from comparison of the analytic conditions for maximization of p and minimization of the mean-square error.

II. MAXIMIZATION OF THE PROBABILITY FIGURE p

A typical servomechanism which shall be considered in these discussions is shown in Fig. 1. The principal function of the device is to close out the error, $\epsilon(t)$. The performance of the system is judged by its success (or failure) to bring the error to zero at one or more specified instants of time or on a time average basis.

The structure and component values of the servomechanism are specified completely except for a number of adjustable parameters $\alpha_1, \alpha_2, \cdots \alpha_n$. Referring to Fig. 2, the error $\epsilon(t)$ results from a non-random input function $\theta_i(t)$ which is zero for $t<0$. It is assumed that as a result of random disturbances arising both inside and outside the system, the error ϵ comprises a systematic (non-random) component ϵ_s, and a

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[3] N. Levinson, *The Wiener RMS Error Criterion in Filter Design and Prediction*, J. Math. and Phys. 25, 261 (1947).

[4] Radiation Laboratory Series, *Theory of Servomechanisms* (McGraw-Hill Book Company, Inc., New York, 1947), pp. 308 *et seq.*

stationary random (noise) component ϵ_N so that

$$\epsilon=\epsilon_s+\epsilon_N. \quad (2)$$

ϵ_N is distributed normally with zero mean and a variance σ^2 which in view of the stationariness of ϵ_N is equal to its mean-square value. These assumptions are found to be generally true in practice.

The object of the design is to assign values to α_i which maximize the probability that at a prescribed instant $t=t_0$, the magnitude of ϵ would be less than a given tolerance L, as shown in Fig. 2. In short, the requirement is that

$$\frac{\partial p_0}{\partial \alpha_i}=0; \quad i=1, 2, 3, \cdots, n \quad (3)$$

where

$$p_0=\text{Prob}\{|\epsilon|_{t=t_0}<L\}. \quad (4)$$

These conditions are subject, of course, to the physical realizability of the system.

As stated, the problem requires that $\alpha_1, \alpha_2\cdots\alpha_n$ be determined on the basis of maximization of p_0. With a normally distributed noise variable the probability p_0 may be written as

$$p_0=\frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\int_{\epsilon_s^0-L}^{\epsilon_s^0+L}\exp\left(-\frac{1}{2}\frac{u^2}{\sigma^2}\right)du, \quad (5)$$

where u is the variable of integration and ϵ_s^0 denotes the value of the systematic error at time $t=t_0$. For any particular type of input θ_i which is selected as a test function, both ϵ_s^0 and σ can be expressed as functions of $\alpha_1, \alpha_2\cdots\alpha_n$. Thus Eq. (5) may be regarded as expressing p_0 as an explicit function of $\alpha_1, \alpha_2\cdots\alpha_n$. To find the equations satisfied by the maximizing values of α_i, it is sufficient to differentiate the right-hand member of (5) partially with respect to the α's and to equate the derivatives to zero. The results of this differentiation are

$$(2\pi)^{\frac{1}{2}}\frac{\partial p_0}{\partial \alpha_i}=\frac{1}{\sigma}\frac{\partial \epsilon_s^0}{\partial \alpha_i}\left\{\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]-\exp\left[-\frac{(\epsilon_s^0-L)^2}{2\sigma^2}\right]\right\}+\frac{\partial\sigma}{\partial\alpha_i}\frac{1}{\sigma^2}\times\int_{\epsilon_s^0-L}^{\epsilon_s^0+L}(u^2/\sigma^2-1)\times\exp\left[-\frac{u^2}{2\sigma^2}\right]du. \quad (6)$$

The coefficient of $\partial\sigma/\partial\alpha_i$ in (6) is integrable and can easily be shown to be equal to

$$\frac{1}{\sigma^2}\int_{\epsilon_s^0-L}^{\epsilon_s^0+L}(u^2/\sigma^2-1)\exp\left[-\frac{u^2}{2\sigma^2}\right]du=\frac{(\epsilon_s^0-L)}{\sigma^2}\exp\left[-\frac{(\epsilon_s^0-L)^2}{2\sigma^2}\right]-\frac{(\epsilon_s^0+L)}{\sigma^2}\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]. \quad (7)$$

Thus, finally, the equations satisfied by the maximizing values of $\alpha_1, \alpha_2\cdots\alpha_n$ take the form

$$\frac{\partial\epsilon_s^0}{\partial\alpha_i}\left\{\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]-\exp\left[-\frac{(\epsilon_s^0-L)}{2\sigma^2}\right]\right\}-\frac{\partial\sigma}{\partial\alpha_i}\left\{\frac{(\epsilon_s^0+L)}{\sigma}\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]-\frac{(\epsilon_s^0-L)}{\sigma}\exp\left[-\frac{(\epsilon_s^0-L)^2}{2\sigma^2}\right]\right\}=0. \quad (8)$$

The complexity of these equations is evident, and their solution can be arrived at only by cut and trial. It will be shown in the next section that under certain circumstances the minimum mean-square error criterion results in the same values for the control coefficients α_i, as would be obtained from solution of (8).

III. COMPARISON OF MAXIMUM PROBABILITY AND MINIMUM MEAN-SQUARE ERROR CRITERIA

In order to make comparisons between the two criteria, an expression similar to (8) will be obtained for the case of the minimum mean-square error criterion. First it should be noted that since the signal is a non-repetitive transient and the error must be minimized at $t=t_0$, the conventional minimum mean-square error criterion cannot be applied without some modifications. It is necessary to regard E^2 not as the average of ϵ^2 over a long period of time, but rather as an average of ϵ^2 over the ensemble. With this understanding, the mean-square value of the

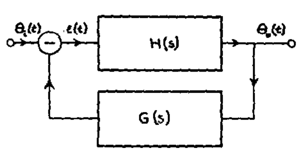

FIG. 1. Schematic diagram of a simple servomechanism.

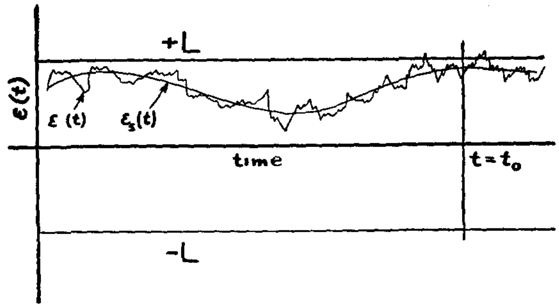

FIG. 2. Error function of a servomechanism.

error takes the form

$$E^2=(\epsilon_s^0)^2+\sigma^2, \tag{9}$$

and the minimizing values of α_i satisfy the equations

$$\frac{\partial E^2}{\partial \alpha_i}=0, \tag{10}$$

that is

$$\frac{\partial \epsilon_s^0}{\partial \alpha_i}\epsilon_s^0+\frac{\partial \sigma}{\partial \alpha_i}\sigma=0;\quad i=1,2,\cdots,n. \tag{11}$$

These equations are the analogs of Eqs. (8) which were obtained on the basis of maximization of p_0. It is the relative simplicity of Eqs. (11), as contrasted to the complexity of Eqs. (8), that makes it possible in many instances to obtain solutions to (11) by purely analytic means.

In order that the solutions of Eqs. (11) be equal to the solutions of (8), it is necessary that the coefficients of $\partial\epsilon_s^0/\partial\alpha_i$ and $\partial\sigma/\partial\alpha_i$ be proportional to each other. That is

$$\frac{\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]-\exp\left[-\frac{(\epsilon_s^0-L)^2}{2\sigma^2}\right]}{\epsilon_s^0}=\frac{\frac{(\epsilon_s^0+L)}{\sigma}\exp\left[-\frac{(\epsilon_s^0+L)^2}{2\sigma^2}\right]-\frac{(\epsilon_s^0-L)}{\sigma}\exp\left[-\frac{(\epsilon_s^0-L)^2}{2\sigma^2}\right]}{\sigma}. \tag{12}$$

After some manipulation, the above relation reduces to the following:

$$\exp\left[-\frac{2\epsilon_s^0L}{\sigma^2}\right]=\frac{1-\frac{\epsilon_s^0L}{\sigma^2}+\frac{(\epsilon_s^0)^2}{\sigma^2}}{1+\frac{\epsilon_s^0L}{\sigma^2}+\frac{(\epsilon_s^0)^2}{\sigma^2}}. \tag{13}$$

Within the limitations of the assumptions made in the statement of the problem, Eq. (13) represents the necessary and sufficient condition for the equivalence of the minimum mean-square error and maximum probability criteria.

IV. DISCUSSION OF RESULTS

An examination of (13) shows that there are only two situations of practical interest where this equation is satisfied approximately over a wide range of values of L, ϵ_s^0, and σ. These conditions are

$$\left.\begin{array}{l}|\epsilon_s^0|\ll\sigma\\(L\epsilon_s^0)^{\frac{1}{2}}\ll\sigma\end{array}\right\} \tag{14}$$

or

$$\begin{array}{l}|\epsilon_s^0|\gg\sigma\\|\epsilon_s^0|\cong L.\end{array} \tag{15}$$

Stated in words, the mean-square error and maximum probability criteria are approximately equivalent when either:

(a) The systematic error as well as the geometric mean of the tolerance L and the systematic error at the time $t=t_0$ are small in comparison with the r.m.s. value of the random error.

(b) The systematic error is large in comparison with the random error, and the tolerance L is approximately equal to the systematic error.

V. CONCLUSION

In the design of that class of servomechanism in which it is desired to close out the error within a prescribed tolerance at a specific instant such as at the termination of a given operation, it is necessary to maximize the probability that the error at that instant be less than the given tolerance. This can be done by adjustment of the dynamic constants of the system in accordance with Eqs. (8). The process is generally quite complicated and must be carried out by cut and trial. Under the general conditions expressed by (13) and in particular by (14) and (15), the resultant design is the same as that which would be obtained by the use of the minimum mean-square error criterion. In view of the relative simplicity of the latter, considerable economy in computational labor can thus be achieved.

Measurements of the Mechanical Properties of Polymer Solutions by Electromagnetic Transducers*

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A concentrated polymer solution is sheared by a rod oscillating axially with a very small amplitude in a closed tube. The rod is driven by a coil in a magnetic field; from electrical measurements on the coil, the mechanical resistance and reactance of the system are calculated by transducer relationships. Equations are given for obtaining the dynamic viscosity, η', and rigidity, G', of the solution. In the case of some polymer solutions, the audiofrequency range covered by the present apparatus falls in the dispersion zone where η' and G' change rapidly with frequency. In other cases the frequency range covered appears to be at the upper end of the dispersion zone; η' is far smaller than the viscosity in steady flow, and G' changes very little with frequency. The advantages and limitations of the method are discussed.

INTRODUCTION

AMONG the diverse phenomena exhibited in the mechanical properties of concentrated polymer solutions, their behavior in small oscillating deformations is of particular interest.[1] At a single frequency of oscillation, the mechanical behavior of such a solution can be simply described by two quantities, the dynamic rigidity and dynamic viscosity. In general, both these quantities are found to be frequency-dependent, and in many cases their dispersion is spread over a wide range of frequencies which cannot be covered by a single experimental method. A variety of methods is required for an adequate study.

Measurements of transverse wave propagation[2] have proved useful for obtaining the rigidity, and under favorable conditions also the viscosity, in the audiofrequency range, provided the mechanical loss factor of the solution is not so great that the waves are too severely damped. The new method described in this paper provides a better determination of viscosity and is applicable for both viscosity and rigidity under some conditions where wave measurements fail owing to extreme damping.

If non-linear effects in either rigidity or viscosity can safely be ignored, as they usually are in treatments of viscoelastic systems,[1,3] both shear and rate of shear must be kept small throughout the cycle of deformation. Fulfilment of this requirement is facilitated by our present apparatus, in which neither shear strain nor stress is measured; the only measurements are of electrical resistance and capacitance, and the viscosity and rigidity are calculated from transducer relationships. In this respect our apparatus differs from that of Philippoff,[4] who employed a somewhat similar device for shearing a solution but calculated the viscosity from measurements of the amplitude of oscillation at mechanical resonance.

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