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Fuzzy Sets and Systems 160 (2009) 3175–3181

FUZZY sets and systems

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Position Paper

Toward extended fuzzy logic—A first step☆

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Received 12 March 2009; received in revised form 3 April 2009; accepted 18 April 2009
Available online 5 May 2009

Abstract

Fuzzy logic adds to bivalent logic an important capability—a capability to reason precisely with imperfect information. Imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague or partially true. In fuzzy logic, results of reasoning are expected to be provably valid, or *p*-valid for short. Extended fuzzy logic adds an equally important capability—a capability to reason imprecisely with imperfect information. This capability comes into play when precise reasoning is infeasible, excessively costly or unneeded. In extended fuzzy logic, *p*-validity of results is desirable but not required. What is admissible is a mode of reasoning which is fuzzily valid, or *f*-valid for short. Actually, much of everyday human reasoning is *f*-valid reasoning.

f-Valid reasoning falls within the province of what may be called unprecisiated fuzzy logic, FLu. FLu is the logic which underlies what is referred to as *f*-geometry. In *f*-geometry, geometric figures are drawn by hand with a spray pen—a miniaturized spray can. In Euclidean geometry, a crisp concept, *C*, corresponds to a fuzzy concept, *f*-*C*, in *f*-geometry. *f*-*C* is referred to as an *f*-transform of *C*, with *C* serving as the prototype of *f*-*C*. *f*-*C* may be interpreted as the result of execution of the instructions: Draw *C* by hand with a spray pen. Thus, in *f*-geometry we have *f*-points, *f*-lines, *f*-triangles, *f*-circles, etc. In addition, we have *f*-transforms of higher-level concepts: *f*-parallel, *f*-similar, *f*-axiom, *f*-definition, *f*-theorem, etc. In *f*-geometry, *p*-valid reasoning does not apply. Basically, *f*-geometry may be viewed as an *f*-transform of Euclidean geometry.

What is important to note is that *f*-valid reasoning based on a realistic model may be more useful than *p*-valid reasoning based on an unrealistic model.

Published by Elsevier B.V.

Keywords: Fuzzy logic; Fuzzy set theory; Fuzzy geometry; Fuzzy theorem

1. From fuzzy logic to extended fuzzy logic—Introduction

The extended fuzzy logic, FLe, is a venture into unchartered territory—a territory in which reasoning and formalisms are quasi-mathematical rather than mathematical. The following is a very brief exposition of some of the basic ideas which underlie FLe. It should be stressed that what follows is just the first step toward construction of extended fuzzy logic and an exploration of its implications and applications.

☆ Research supported in part by ONR N00014-02-1-0294, BT Grant CT1080028046, Omron Grant, Tekes Grant, Chevron Texaco Grant, The Ministry of Communications and Information Technology of Azerbaijan and the BISC Program of UC Berkeley.

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[1] To my late mentors: Richard Bellman (systems analysis), Herbert Robbins (probability theory) and Stephen Kleene (logic).

0165-0114/$ - see front matter Published by Elsevier B.V.
doi:10.1016/j.fss.2009.04.009

Science deals not with reality but with models of reality. In large measure, scientific progress is driven by a quest for better models of reality. In constructing better models of reality, a problem that has to be faced is that as the complexity of a system, S, increases, it becomes increasingly difficult to construct a model, $M(S)$, which is both cointensive, that is, close-fitting, and precise. This applies, in particular, to systems in which human judgment, perceptions and emotions play a prominent role. Economic systems, legal systems and political systems are cases in point.

As the complexity of a system increases further, a point is reached at which construction of a model which is both cointensive and precise is not merely difficult—it is impossible. It is at this point that extended fuzzy logic comes into play.

Actually, extended fuzzy logic is not the only formalism that comes into play at this point. The issue of what to do when an exact solution cannot be found or is excessively costly is associated with a vast literature. Prominent in this literature are various approximation theories [2], theories centered on bounded rationality [11], qualitative reasoning [14], commonsense reasoning [4,6] and theories of argumentation [12]. Extended fuzzy logic differs from these and related theories both in spirit and in substance. The difference will become apparent in Section 2, in which the so-called *f*-geometry is used as an illustration.

To develop an understanding of extended fuzzy logic, FLe, it is expedient to start with the following definition of fuzzy logic, FL. Fuzzy logic is a precise conceptual system of reasoning, deduction and computation in which the objects of discourse and analysis are, or are allowed to be, associated with imperfect information. Imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague or partially true. In fuzzy logic, the results of reasoning, deduction and computation are expected to be provably valid (*p*-valid) within the conceptual structure of fuzzy logic.

There are many misconceptions about fuzzy logic. The principal misconception is that fuzzy logic is fuzzy. The stated definition underscores that fuzzy logic is precise. In fuzzy logic precision is achieved through association of fuzzy sets with membership functions and, more generally, association of granules with generalized constraints [17]. What this implies is that fuzzy logic is what may be called precisiated logic.

At this point, a key idea comes into play. The idea is that of constructing a fuzzy logic, FLu, which, in contrast to FL, is unprecisiated. What this means is that in FLu membership functions and generalized constraints are not specified, and are a matter of perception rather than measurement. To stress the contrast between FL and FLu, FL may be written as FLp, with *p* standing for precisiated.

A question which arises is: What is the point of constructing FLu—a logic in which provable validity is off the table? But what is not off the table is what may be called fuzzy validity, or *f*-validity for short. As will be shown in Section 2, a model of FLu is *f*-geometry—a geometry in which figures are drawn by hand with a spray pen, without the use of a ruler or compass. Actually, everyday human reasoning is preponderantly *f*-valid reasoning. Humans have a remarkable capability to perform a wide variety of physical and mental tasks without any measurements and any computations. In this context, *f*-valid reasoning is perception-based. In FLu, there are no formal definitions, theorems or *p*-valid proofs.

The concept of unprecisiated fuzzy logic provides a basis for the concept of extended fuzzy logic, FLe. More specifically, FLe is the result of adding FLu to FL(FLp),

$$\text{FLe} = \text{FL} + \text{FLu},$$

with FLu playing the role of an extension of, or addendum to, FL.

Expressing FLe as the sum of FL and FLu has important implications. First, to construct a definition of FLe it is sufficient to delete the word "precise" from the definition of FL. With this deletion, the definition of FLe reads:

Extended fuzzy logic, FLe, is a conceptual system of reasoning, deduction and computation in which the objects of discourse and analysis are, or are allowed to be, associated with imperfect information. Imperfect information is information which in one or more respects is imprecise, uncertain, incomplete, unreliable, vague or partially true. In extended fuzzy logic, the result of reasoning, deduction or computation is not expected to be provably valid.

Second, and more importantly, *f*-valid reasoning is not admissible in FL, but is admissible in FLe when *p*-valid reasoning is infeasible, carries an excessively high cost or is unneeded. In many realistic settings, this is the norm rather than exception. The following very simple example is a case in point.

I hail a taxi and ask the driver to take me to address A. There are two versions: (a) I ask the driver to take me to A the shortest way; and (b) I ask the driver to take me to A the fastest way. Based on his/her experience, the driver chooses route (a) for (a) and route (b) for (b).

In version (a) if there is a map of the area it is possible to construct the shortest way to A. This would be a *p*-valid solution. Thus, for version (a) there exists a *p*-valid solution but the driver's choice of route (a) may be viewed as an *f*-valid solution which in some sense is good enough.

In version (b), it is not possible to construct a cointensive model of the system and hence it is not possible to construct a *p*-valid solution. The problem is rooted in uncertainties related to traffic conditions, timing of lights, etc. In fact, if the driver had asked me to define what I mean by "the fastest way," I could not come up with an answer to his/her question. Thus, in version (b) there exists an *f*-valid solution, but a *p*-valid solution does not exist.

Basically, extended fuzzy logic, FLe, results from lowering of standards of cointension and precision in fuzzy logic, FL. In effect, extended fuzzy logic adds to fuzzy logic a capability to deal imprecisely with imperfect information when precision is infeasible, carries a high cost or is unneeded. This capability is a necessity when repeated attempts at constructing a theory which is both realistic and precise fail to achieve success. Cases in point are the theories of rationality, causality and decision-making under second order uncertainty, that is, uncertainty about uncertainty.

A useful analogy is the following. In bivalent logic, the writing/drawing instrument is a ballpoint pen. In fuzzy logic, the writing/drawing instrument is a spray pen—a miniature spray can—with an adjustable, precisely specified spray pattern and a white marker for the centroid of the spray pattern, with the marker serving the purpose of precisiation when it is needed. Such a pen will be referred to as precisiated. In unprecisiated fuzzy logic, the spray pen has an adjustable spray pattern and a white marker which are not precisiated. In extended fuzzy logic, there are two spray pens—a precisiated spray pen and an unprecisiated spray pen.

In summary, there are three principal rationales for the use of extended fuzzy logic. First, when a *p*-valid solution is infeasible. Second, when a *p*-valid solution carries an excessively high cost; and third, when there is no need for a *p*-valid solution, that is, when an *f*-valid solution is good enough. In much of everyday human reasoning, it is the third rationale that is preponderant.

There is an important point to be made. *f*-Validity is a fuzzy concept and hence is a matter of degree. When a chain of reasoning leads to a conclusion, a natural question is: What is the possibly fuzzy degree of validity, call it the validity index, of the conclusion? In most applications involving *f*-valid reasoning a high validity index is a desideratum. How can it be achieved? Achievement of a high validity index is one of the principal objectives of extended fuzzy logic. It should be noted that in many realistic settings, the question of whether or not a conclusion has a high validity index may be a matter of argumentation.

To take a step toward construction of modes of reasoning which lead to conclusions which are associated with high validity indices, it is expedient to go back to the origin of logical reasoning—Euclidian geometry. This is what is done in the following.

2. *f*-Geometry and *f*-transformation

In the world of Euclidean geometry, Weg, the drawing instruments are: ruler, compass and ballpoint pen. The underlying logic is the familiar bivalent, Aristotelian logic. In the world of *f*-geometry, Wfg, the only drawing instrument is an unprecisiated spray pen, and drawing is done by hand. Figures in Wfg are fuzzy in appearance. In *f*-geometry, the underlying logic is unprecisiated fuzzy logic, FLu. *f*-Geometry differs both in spirit and in substance from Poston's fuzzy geometry [8], coarse geometry [9], fuzzy geometry of Rosenfeld [10], fuzzy geometry of Buckley and Eslami [1], fuzzy geometry of Mayburov [5], and fuzzy geometry of Tzafestas [13]. The underlying logic in these fuzzy geometries is FL(FLp).

The counterpart of a crisp concept, *C*, in Weg, is a fuzzy concept, *f*-*C* or, when more convenient, $^*C$, in Wfg (Fig. 1). *f*-*C* is referred to as an *f*-transform of *C*, with *C* playing the role of the prototype of *f*-*C*. It is helpful to visualize a fuzzy transform of *C* as the result of execution of the instruction: Draw *C* by hand with a spray pen. Note that there is no formal definition *f*-transformation.

For example, the *f*-transform of a point is an *f*-point, the *f*-transform of a line is an *f*-line, the *f*-transform of a triangle is an *f*-triangle, the *f*-transform of a circle is an *f*-circle and the *f*-transform of parallel is *f*-parallel (Fig. 2).

Note that *f*-transformation is one-to-many. *f*-Transformation may be applied to relations. Thus, in Wfg we have the concepts of *f*-parallel, *f*-similar, *f*-congruent, etc. Furthermore, *f*-transformation may be applied to higher-level concepts, e.g., axiom, definition, principle, proof, theorem, truth, etc. In addition, *f*-transformation may be applied to concepts drawn from fields other than *f*-geometry. Examples: *f*-convex, *f*-linear, *f*-stable, etc. Of particular importance in *f*-geometry is the concept of *f*-theorem [15].

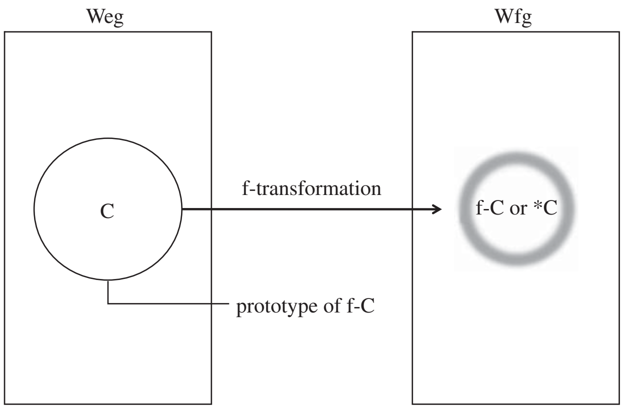


Fig. 1. *f*-Transformation and *f*-geometry. Note that fuzzy figures, as shown, are not hand drawn. They should be visualized as hand drawn figures.

Fig. 2. Examples of *f*-transformation.

The cointension of *f*-*C* is a qualitative measure of the proximity of *f*-*C* to its prototype, *C*. A fuzzy transform, *f*-*C*, is cointensive if its cointension is high. Unless stated to the contrary, *f*-transforms are assumed to be cointensive. The concept of *f*-transform is distinct from the concept of fuzzy transform (Perfilieva transform) of Perfilieva [7]. In summary, *f*-geometry may be viewed as the result of application of *f*-transformation to Euclidean geometry.

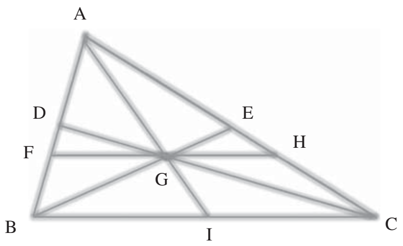
A key idea in *f*-geometry is the following: if *C* is *p*-valid then its *f*-transform, *f*-*C*, is *f*-valid with a high validity index. As a simple example, consider the definition, *D*, of parallelism in Euclidean geometry.

D: Two lines are parallel if for any transversal that cuts the lines the corresponding angles are congruent.

f-transform of this definition reads:

f-*D*: Two *f*-lines are *f*-parallel if for any *f*-transversal that cuts the lines the corresponding *f*-angles are *f*-congruent.

Similarly, in Euclidean geometry, two triangles are similar if the corresponding angles are congruent. Correspondingly, in *f*-geometry two *f*-triangles are *f*-similar if the corresponding angles are *f*-congruent.



D, E are f-midpoints
DE is f-parallel to BC
FH is f-parallel to BC
AGI is an f-line passing through f-point G
f-triangles EGH and EBC are f-similar
f-triangles DFG and DBC are f-similar
f-proportionality of corresponding sides of f-triangles implies that G is f-midpoint of FH
G is f-midpoint of FH implies that I is f- midpoint of BC
I is f-midpoint of BC implies that the f- medians are f-concurrent

Fig. 3. *f*-Proof of the *f*-theorem: *f*-midpoints of an *f*-triangle are *f*-concurrent.

An *f*-theorem in *f*-geometry is an *f*-transform of a theorem in Euclidean geometry. As a simple example, an elementary theorem, *T*, in Euclidean geometry is:

T: the medians of a triangle are concurrent.

A corresponding theorem, *f-T*, in *f*-geometry is:

f-T: the *f*-medians of an *f*-triangle are *f*-concurrent.

An important *f*-principle in *f*-geometry, referred to as the validation principle, is the following. Let p be a p-valid conclusion drawn from a chain of premises $p_1, \ldots, p_n$. Then, using the star notation, $^*p$ is an *f*-valid conclusion drawn from $^*p_1, \ldots, {}^*p_n$, and $^*p$ has a high validity index. It is this principle that is employed to derive *f*-valid conclusions from a collection of *f*-premises. As a very simple illustration, consider two triangles *A* and *B*. In Euclidean geometry, if *A* and *B* are similar then the corresponding sides are in proportion. The validation principle leads to the following assertion in *f*-geometry. If *A* and *B* are *f*-similar *f*-triangles then the corresponding sides are in *f*-proportion.

In *f*-geometry, an *f*-proof may be (a) empirical or (b) logical. An empirical *f*-proof involves experimentation. Consistent with the validation principle, a logical *f*-proof is an *f*-transform of a proof in Euclidean geometry. As an illustration, consider the *f*-theorem:

f-T: the *f*-medians of an *f*-triangle are *f*-concurrent.

With reference to Fig. 3, the logical *f*-proof of this theorem follows at once from the property of *f*-similar triangles.

The *f*-theorem and its *f*-proof are *f*-transforms of their counterparts in Euclidean geometry. But what is important to note is that the *f*-theorem and its *f*-proof could be arrived at without any reference to their counterparts in Euclidean geometry. This suggests an intriguing possibility of constructing, in various fields, independently arrived at systems of *f*-concepts, *f*-definitions, *f*-theorems, *f*-proofs and, more generally, *f*-reasoning and *f*-computation. In the conceptual world of such systems, *p*-validity has no place.

As was alluded to earlier, the concept of *f*-transformation is not limited to Euclidean geometry—it has broad applicability. *f*-Transformation may be applied to concepts, definitions and theorems drawn from various fields. As an elementary example, consider the definition of a convex set, *A*, in a linear vector space, *U*.

D: A is a convex set in *U* if for any points x and y in *A* every point in the segment xy is in *A*. The *f*-transform of this definition is the definition of an *f*-convex set, *f-A*. Specifically,

f-D: *f-A* is an *f*-convex set in *U* if for any *f*-points x and y in *f-A* every *f*-point in the *f*-segment xy is in *f-A*.

An elementary property of convex sets is:

T: if *A* and *B* are convex sets, so is their f-intersection $A \cap B$.

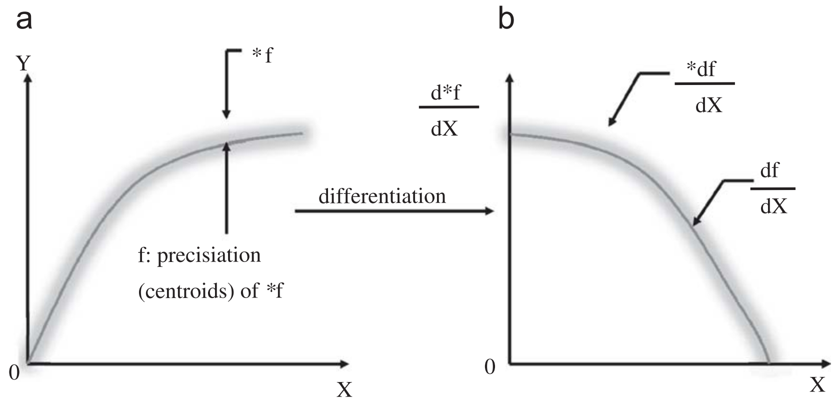


Fig. 4. *f*-Transform of the derivative of an *f*-transform.

An *f*-transform of *T* reads:

f-*T*: if *A* and *B* are *f*-convex sets, so is their *f*-intersection *f*-$A \cap f$-B.

More generally,

T: if *A* and *B* are convex fuzzy sets, so is their intersection.

Applying *f*-transformation to *T*, we obtain the theorem:

f-*T*: if *A* and *B* are *f*-convex fuzzy sets, so is their *f*-intersection.

A basic problem which arises in computation of *f*-transforms is the following. Let *g* be a function, a functional or an operator. Using the star notation, let an *f*-transform, $^*C$, be an argument of *g*. The problem is that of computing $g(^*C)$. Generally, computing $g(^*C)$ is not a trivial problem.

An *f*-valid approximation to $g(^*C)$ may be derived through application of an *f*-principle which is referred to as precisiation/imprecisiation principle or P/I principle, for short [16]. More specifically, the principle may be expressed as

$$g(^*C)^* = {}^*g(C)$$

where $^*=$should be read as approximately equal. In words, $g(^*C)$ is approximately equal to the *f*-transform of $g(C)$.

As an illustration, if *g* is the operation of differentiation and $^*C$ is an *f*-function, $^*f$, shown in Fig. 4a, then the *f*-derivative of this function is an *f*-function shown in Fig. 4b.

If *C* is a real number and $^*C$ is approximately *C*, then the P/I principle asserts that $g(^*C)$ is approximately equal to approximately $g(C)$. More generally, if *C* is a function from reals to reals, $^*C$ is the fuzzy graph of *C* [16] and *g* is the operation of differentiation, then the derivative of the fuzzy graph of *C* is approximately equal to the fuzzy graph of the derivative of *C*. An example is shown in Fig. 5.

In one guise or another, the P/I principle is widely used in science and engineering. What should be a matter of concern, however, is that it is common practice to present the results of an analysis in which the principle is employed without a qualification to the effect that the results are *f*-valid rather than *p*-valid, and that there is no guarantee that the validity index of results is high.

3. Concluding remark

In large measure, extended fuzzy logic is perception-based rather than measurement-based. Perceptions are intrinsically imprecise, reflecting the bounded ability of human sensory organs, and ultimately the brain, to resolve detail and store information. The intrinsic imprecision of perceptions underlies the intrinsic imprecision of a major component

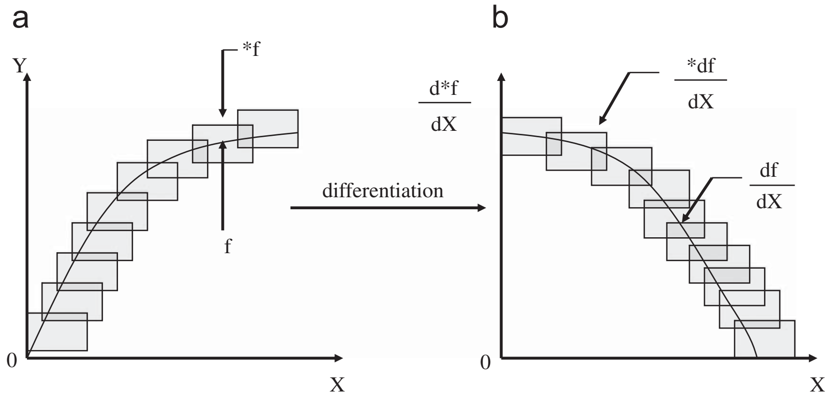


Fig. 5. *f*-Transform of the derivative of a fuzzy graph.

of extended fuzzy logic—the unprecisiated fuzzy logic. As was alluded to earlier, reasoning in unprecisiated fuzzy logic—*f*-valid reasoning—is quasi-mathematical rather than mathematical. This is what sets unprecisiated fuzzy logic apart from other logical systems, including precisiated fuzzy logic, FL.

The importance of extended fuzzy logic derives from the fact that it adds to fuzzy logic an essential capability—the capability to deal with unprecisiated imperfect information. In many realistic settings, decision-relevant information falls into this category. In addition, what should be underscored is that in dealing with many real-world problems an *f*-valid solution based on a realistic model may be more useful than a *p*-valid solution based on an unrealistic model.

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