COMPUTATION WITH IMPRECISE PROBABILITIES

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KEY POINTS

- Information analysis is a portal to decision analysis.
- In most realistic settings, decision-relevant information is imperfect.
Imperfect information: imprecise and/or uncertain and/or incomplete and/or partially true.

Second-order uncertainty = uncertainty about uncertainty = uncertainty^2.

Generally, imperfect information is uncertain^2.

Imprecise probabilities, fuzzy probabilities, fuzzy sets of Type 2 are uncertain^2.

Decision analysis under uncertainty^2 is uncharted territory.
IMPRECISE PROBABILITIES
BASIC MODALITIES

- *imprecise probabilities*
  - *data-based*
  - *perception-based*

- Walley (1991) et al
- Described in NL Computing with Words (CW)
THE CW-BASED APPROACH TO COMPUTATION WITH IMPRECISE PROBABILITIES

- probabilities
- utilities
- relations
- events
- constraints
- goals
- ...

- Information about the embedded constituents of S is, or is allowed to be, imperfect.
- Information about the embedded constituents of S is, or is allowed to be, described in natural language.
I have to fly from A to D, and would like to get there as soon as possible.

I have two choices: (a) fly to D with a connection in B; or (b) fly to D with a connection in C.

if I choose (a), I will arrive in D at time $t_1$

if I choose (b), I will arrive in D at time $t_2$

$t_1$ is earlier than $t_2$

therefore, I should choose (a)?
A SIMPLE EXAMPLE OF COMPUTATION

A small glass jar contains about 20 balls of various sizes. Most are small. A few are large. What is the average size of balls?

Solution

Population = (b₁, ..., bₙ)

dᵢ = diameter of bᵢ, i=1, ..., N
A glass jar contains about 20 balls. $N$ is $20$

Most are small:
$\frac{\mu_{\text{small}}(d_1)+\ldots+\mu_{\text{small}}(d_N)}{N}$ is most

A few are large:
$\mu_{\text{large}}(d_1)+\ldots+\mu_{\text{large}}(d_N)$ is few

What is the average size of balls?

$\frac{d_1+\ldots+d_N}{N}$ is $A$
Continued

\[ N \text{ is } *20 \]

\[
\left( \mu_{\text{small}}(d_1) + \ldots + \mu_{\text{small}}(d_N) \right) / N \text{ is most}
\]

\[
\mu_{\text{large}}(d_1) + \ldots + \mu_{\text{large}}(d_N) \text{ is few}
\]

\[
(d_1 + \ldots + d_N) / N \text{ is ?A}
\]

- Solution reduces to generalized constraint propagation
Science deals not with reality but with models of reality. In large measure, scientific progress is driven by a quest for better models of reality.

The real world is pervaded with various forms of imprecision and uncertainty. To construct better models of reality, it is essential to develop a better understanding of how to deal with different forms of imprecision and uncertainty.
PREAMBLE

- An imprecise probability distribution is an instance of second-order uncertainty, that is, uncertainty about uncertainty.
- Computation with imprecise probabilities is not an academic exercise—it is a bridge to reality. In the real world, imprecise probabilities are the norm rather than exception.
- In large measure, real-world probabilities are perceptions of likelihood. Perceptions are intrinsically imprecise. Imprecision of perceptions entails imprecision of probabilities.
NECESSITY OF IMPRECISION

- Can you explain to me the meaning of “Speed limit is 65 mph?”
- No imprecise numbers and no probabilities are allowed
- Imprecise numbers are allowed. No probabilities are allowed.
- Imprecise numbers are allowed. Precise probabilities are allowed.
- Imprecise numbers are allowed. Imprecise probabilities are allowed.
NECESSITY OF IMPRECISION

- Can you precisiate the meaning of “arthritis”?
- Can you precisiate the meaning of “recession”?
- Can you precisiate the meaning of “beyond reasonable doubt”?
- Can you precisiate the meaning of “causality”?
- Can you precisiate the meaning of “randomness”? 
NECESSITY OF IMPRECISION

- Can you precisiate the meaning of “imprecise probability?”
- Can you precisiate the meaning of “risk aversion?”
IMPRECISION OF MEANING

- Imprecision of meaning = elasticity of meaning
- Elasticity of meaning = fuzziness of meaning

Example: middle-aged
CONTINUED


- In the mainstream literature on imprecise probabilities, imprecise probabilities are dealt within the conceptual framework of standard probability theory.
What is widely unrecognized is that standard probability theory, call it PT, has a serious limitation. More specifically, PT is based on bivalent logic—a logic which is intolerant of imprecision and does not admit shades of truth and possibility. As a consequence, the conceptual framework of PT is not the right framework for dealing with imprecision and, more particularly, with imprecision of information which is described in natural language.
The approach to computation with imprecise probabilities which is described in the following is a radical departure from the mainstream literature.
Its principal distinguishing features are: (a) imprecise probabilities are dealt with not in isolation, as in the mainstream approaches, but in an environment of imprecision of events, relations and constraints; (b) imprecise probabilities are assumed to be described in a natural language. This assumption is consistent with the fact that a natural language is basically a system for describing perceptions.
MODALITIES OF PROBABILITY

probability

- objective
  - measurement-based
- subjective
  - perception-based
  - NL-based
EXAMPLES

1. $X$ is a real-valued random variable. What is known about $X$ is: (a) usually $X$ is much larger than approximately $a$; and (b) usually $X$ is much smaller than approximately $b$, with $a < b$. What is the expected value of $X$?

2. $X$ is a real-valued random variable. What is known is that $\text{Prob}(X \text{ is small})$ is low; $\text{Prob}(X \text{ is medium})$ is high; and $\text{Prob}(X \text{ is large})$ is low. What is the expected value of $X$?
3. A box contains approximately twenty balls of various sizes. Most are small. There are many more small balls than large balls. What is the probability that a ball drawn at random is neither large nor small?
EXAMPLES

4. I am checking-in for my flight. I ask the ticket agent: What is the probability that my flight will be delayed. He tells me: Usually most flights leave on time. Rarely most flights are delayed. How should I use this information to assess the probability that my flight may be delayed?
BASIC APPROACH

- To compute with information described in natural language we employ the formalism of Computing with Words (CW) (Zadeh 1999) or, more generally, NL-Computation (Zadeh 2006).

- The formalism of Computing with Words, in application to computation with information described in a natural language, involves two basic steps: (a) precisiation of meaning of propositions expressed in natural language; and (b) computation with precisiated propositions.
Precisiation of meaning is achieved through the use of generalized-constraint-based semantics, or GCS for short.

The concept of a generalized constraint is the centerpiece of GCS. Importantly, generalized constraints, in contrast to standard constraints, have elasticity. What this implies is that in GCS everything is or is allowed to be graduated, that is, be a matter of degree.
The concept of a fuzzy set

A fuzzy set is a class with a fuzzy boundary.

A fuzzy set is precisiated through graduation, that is, through association with a scale of grades of membership. Thus, membership in a fuzzy set is a matter of degree. A fuzzy set is not a set.
A clarification is in order. Consider a concatenation of two words, A and B, with A modifying B, e.g. A is an adjective and B is a noun. Usually, A plays the role of an s-modifier (specializer), that is, a modifier which specializes B in the sense that AB is a subset of B, as in convex set. In some instances, however, A plays the role of a g-modifier (generalizer), that is, a modifier which generalizes B. An example is a random set.
In this sense, fuzzy in fuzzy set and fuzzy logic is a generalizer, but is frequently misinterpreted as a specializer. Interestingly, g-modification implies that a set is a fuzzy set but a fuzzy set is not a set. Similarly, a logic is fuzzy logic but fuzzy logic is not logic. Similarly, a probability is fuzzy probability, but fuzzy probability is not probability.
GRANULATION

- In GCS everything is or is allowed to be granulated. Granulation involves partitioning of an object into granules, with a granule being a clump of elements drawn together by indistinguishability, equivalence, similarity, proximity or functionality. A granule, G, is precisiated through association with G of a generalized constraint.
SINGULAR AND GRANULAR VALUES

A

granular value of X

singular value of X

universe of discourse

singular

granular

unemployment

7.3%

high

102.5

very high

160/80

high

temperature

blood pressure
GRANULAR PROBABILITIES

- A granular value is precisiated via a generalized constraint.
- A granular value of probability is a granular probability.

Examples:
- Likely, not likely, unlikely, very likely, very unlikely, usually, low, high, etc.

- The concept of granular probability is more general than the concept of fuzzy probability.
Granulation may be viewed as a form of summarization/information compression.

Humans employ qualitative granulation to deal with imprecision, uncertainty and complexity.
Precisiation of meaning is a prerequisite to computation with CW-based imprecise probabilities. A brief exposition of the basic concepts and techniques which relate to precisiation of meaning is presented in the following.
In one form or another, precisiation of meaning has always played an important role in science. Mathematics is a quintessential example of what may be called a meaning precisiation language system.

Note: A language system differs from a language in that in addition to descriptive capability it has a deductive capability. For example, probability theory may be viewed as a precisiation language system so is Prologue. A natural language is a language rather than a language system.
Precisiation of meaning has direct relevance to mechanization of natural language understanding. For this reason, precisiation of meaning is an issue that is certain to grow in visibility and importance as we move further into the age of machine intelligence and automated reasoning.
Semantic imprecision of natural languages is a very basic characteristic—a characteristic which is rooted in imprecision of perceptions. Basically, a natural language is a system for describing perceptions. Perceptions are imprecise. Imprecision of perceptions entails semantic imprecision of natural languages.
SEMIANTIC IMPRECISION (EXPLICIT)
EXAMPLES

WORDS/CONCEPTS
- Recession
- Civil war
- Very slow
- Honesty
- Arthritis
- Random variable
- Cluster
- Stationarity

PROPOSITIONS
- It is likely to be warm tomorrow.
- It is very unlikely that there will be a significant decrease in the price of oil in the near future.
SEMANTIC IMPRECISION (IMPLICIT)

EXAMPLES

- Speed limit is 100 kmh
- Checkout time is 1 pm
NECESSITY OF IMPRECISION

- Can you explain to me the meaning of “Speed limit is 100 kmh?”
- No imprecise numbers and no probabilities are allowed
- Imprecise numbers are allowed. No probabilities are allowed.
- Imprecise numbers are allowed. Precise probabilities are allowed.
- Imprecise numbers are allowed. Imprecise probabilities are allowed.
NECESSITY OF IMPRECISION

- Can you precisiate the meaning of “arthritis”? 
- Can you precisiate the meaning of “recession”? 
- Can you precisiate the meaning of “beyond reasonable doubt”? 
- Can you precisiate the meaning of “causality”? 
- Can you precisiate the meaning of “near”? 
Traditional approaches to semantics of natural languages, among them truth-conditional semantics, possible-world semantics and Montague semantics, do not address the issue of semantic imprecision of natural languages. The issue is not addressed because the conceptual framework of traditional approaches is not the right framework for dealing with semantic imprecision. In traditional approaches, elasticity of meaning is not dealt with.
There is a need for new direction. It is this need that motivates generalized-constraint-based semantics of natural languages, or GCS for short. As an issue, semantic imprecision has a position of centrality in GCS.

The point of departure in GCS is the concept of precisiation of meaning—a concept which goes beyond the familiar concept of representation of meaning.
**SEMANTICS**

**Meaning**
- Representation
- Precisiation

**r-semantics**
- Traditional

**p-semantics**
- Nontraditional

Lily is young → Age(Lily) is young

*Diagram showing the relationship between representation and precisiation.*
most Swedes are tall \[ \text{representation} \]
Count(tall.Swedes/Swedes) is most.

\[ \text{precisiation} \]
Count(tall.Swedes/Swedes)

\[ \sum \text{Count(tall.Swedes/Swedes)} \]
THE CONCEPTS OF PRECISION AND COINTENSIVE PRECISION
The concept of precision has a position of centrality in scientific theories. And yet, there are some important aspects of this concept which have not been adequately treated in the literature. One such aspect relates to the distinction between precision of value (v-precision) and precision of meaning (m-precision).

The same distinction applies to imprecision, precisiation and imprecisiation.
• $p$: $X$ is in the interval $[a, b]$. $a$ and $b$ are precisely defined real numbers
• $p$ is v-imprecise and m-precise

• $p$: $X$ is a Gaussian random variable with mean $m$ and variance $\sigma^2$. $m$ and $\sigma^2$ are precisely defined real numbers
• $p$ is v-imprecise and m-precise
A proposition, predicate, query or command may be precisiated or imprecisiated.

Examples:

- "m-precisiation" example:
  - "Lily is 25" → "Lily is young"

- "v-imprecisiation" example:
  - "young" → "m-precisiation" example:
  - "Lily is 25" → "Lily is young"

- Data compression and summarization are instances of imprecisiation.
MODALITIES OF m-PRECISIATION

- **m-precisiation**
  - **mh-precisiation**
    - Human-oriented
  - **mm-precisiation**
    - Machine-oriented
      - Mathematically well-defined

**Example:** bear market
- mh-precisiation: declining stock market with expectation of further decline

- mm-precisiation: 30 percent decline after 50 days, or a 13 percent decline after 145 days. (Robert Shuster)
**BASIC CONCEPTS**

- **precisiand** = model of meaning
- **precisiation** ~ modelization
- **intension** = attribute-based meaning
- **cointension** = measure of closeness of meanings
  = measure of closeness of model

- **precisiation** = translation into a precisiation language system

\[ p: \text{object of precisiation} \rightarrow \text{precisiation} \rightarrow \text{precisiand} \]

\[ p^*: \text{result of precisiation} \]

\[ \text{cointension} \]
CONTINUED

precisiend → precisiation → precisiation

p

cointension

p

imprecisiend → imprecisiation → imprecisiation

imprecisiend

*p

cointension

*p
MM-PRECISIATION OF “approximately a,” *a
(MODELS OF MEANING OF *a)

Bivalent Logic

number

interval

probability

It is a common practice to ignore imprecision, treating what is imprecise as if it were precise.

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Fuzzy Logic: Bivalent Logic + …

Fuzzy interval

Fuzzy interval type 2

Fuzzy probability

Fuzzy logic has a much higher expressive power than bivalent logic.
GOODNESS OF MODEL OF MEANING

goodness of model = (cointension, computational complexity)

*a: approximately a

two numbers

four numbers

one number

cointension

computational complexity

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Recession (mh-precisian):  
- A period of an economic contraction, sometimes limited in scope or duration.

Recession (mm-precisian):  
- A recession is a decline in a country's gross domestic product (GDP), or negative real economic growth, for two or more successive quarters of a year.
PRECISIATION IN COMMUNICATION

- mm-precisiation is desirable but not mandatory
- mm-precisiation is mandatory

Humans can understand unprecisiated natural language. Machines cannot.

Scientific progress: mh-precisiation → mm-precisiation
Recipient: I understand what you sent, but could you precisiate what you mean, using ... (restrictions)?

Sender: (a) I will be pleased to do so
(b) sorry, it is your problem
In mechanization of natural language understanding, the precisiator is the machine.

In most applications of fuzzy logic, the precisiator is the human. In this case, context-dependence is not a problem. As a consequence, precisiation is a much simpler function.
Control Rules:

1. If (speed is low) and (shift is high) then (-3)
2. If (speed is high) and (shift is low) then (+3)
3. If (throt is low) and (speed is high) then (+3)
4. If (throt is low) and (speed is low) then (+1)
5. If (throt is high) and (speed is high) then (-1)
6. If (throt is high) and (speed is low) then (-3)
Declarative graduation/standardization of normal temperature, fever and high fever
Elicitation/exemplification exploits the remarkable human capability to rank-order perceptions. Example: precisiation of Vera is middle-aged.

Assume that A tells B that Vera is middle-aged. B can elicit A’s meaning of middle-aged by asking A to mark on the scale [0, 1] the degree to which a particular age, say 43, fits A’s meaning of middle-aged. The process is repeated for various values of Age. Eventually, the collected data are employed to approximate to A’s meaning of middle-aged by a trapezoidal membership function. In the case of fuzzy sets of Type 2, the mark is a fuzzy point.
ELICITED MEANING OF MIDDLE-AGED

Model of middle-aged

μ

0 40 45 55 60

0 0.8 1

definitely not middle-age
definitely middle-age
definitely not middle-age

43
v-IMPRECISIATION

- Imperative (forced)
- Intentional (deliberate)

imperative: value is not known precisely
intentional: value need not be known precisely

- data compression and summarization are instances of v-imprecisiation
THE FUZZY LOGIC GAMBIT

Fuzzy logic gambit = v-imprescision followed by mm-precision
Most applications of fuzzy logic in the realm of consumer products employ the fuzzy logic gambit.

Basically, the fuzzy logic gambit exploits a tolerance for imprecision to achieve reduction in cost.
THE CONCEPT OF COINTENSIVE PRECISIATION

- $m$-precisiation of a concept or proposition, $p$, is cointensive if $p^*$ is cointensive with $p$.

Example: bear market

We classify a bear market as a 30 percent decline after 50 days, or a 13 percent decline after 145 days. (Robert Shuster)

This definition is clearly not cointensive
EXAMPLE: IMPACT FACTOR

- \( A = \text{citations in 1992 to articles published in 1987-91} \)

- \( B = \text{articles published in 1987-91} \)

- \( C = \frac{A}{B} = \text{five-year impact factor} \)

- Is the definition of impact factor cointensive?
Basic question

- Given a proposition, p, how can p be cointesively mm-precisiated?

Key idea

- In generalized-constraint-based semantics, mm-precisiation is carried out through the use of the concept of a generalized constraint.
- What is a generalized constraint?
THE CONCEPT OF A GENERALIZED CONSTRAINT

A BRIEF INTRODUCTION
The concept of a generalized constraint is the centerpiece of generalized-constraint-based semantics.

In scientific theories, representation of constraints is generally oversimplified. Oversimplification of constraints is a necessity because existing constraint definition languages have a very limited expressive power.
The concept of a generalized constraint is intended to provide a basis for construction of a maximally expressive meaning precisiation language for natural languages.

Generalized constraints have elasticity.

Elasticity of generalized constraints is a reflection of elasticity of meaning of words in a natural language.
GENERALIZED CONSTRAINT (Zadeh 1986)

- Bivalent constraint (hard, inelastic, categorical:)
  \[ X \in C \]
  constraining bivalent relation

- Generalized constraint on \( X \): \( GC(X) \)  (elastic)

  \[ GC(X): X \text{ isr } R \]
  constraining non-bivalent (fuzzy) relation

  index of modality (defines semantics)

  constrained variable

  \( r: \epsilon = \leq \geq \subset \ldots \text{ blank } | p | v | u | rs | fg | ps | \ldots \)

  bivalent

  primary

- open \( GC(X) \): \( X \) is free (\( GC(X) \) is a predicate)

- closed \( GC(X) \): \( X \) is instantiated (\( GC(X) \) is a proposition)
**GENERALIZED CONSTRAINT—MODALITY \( r \)**

- **\( r = \)** equality constraint: \( X = R \) is abbreviation of \( X \) is \( = R \)
- **\( r \leq \)** inequality constraint: \( X \leq R \)
- **\( r \subset \)** subsethood constraint: \( X \subset R \)
- **\( r: \text{blank} \)** possibilistic constraint; \( X \) is \( R \); \( R \) is the possibility distribution of \( X \)
- **\( r: v \)** veristic constraint; \( X \) is \( v R \); \( R \) is the verity distribution of \( X \)
- **\( r: p \)** probabilistic constraint; \( X \) is \( p R \); \( R \) is the probability distribution of \( X \)

**Standard constraints:** bivalent possibilistic, bivalent veristic and probabilistic
CONTINUED

\[ r: \text{rs} \quad \text{random set constraint; } X \text{ isrs } R; R \text{ is the set-valued probability distribution of } X \]

\[ r: \text{fg} \quad \text{fuzzy graph constraint; } X \text{ isfg } R; X \text{ is a function and } R \text{ is its fuzzy graph} \]

\[ r: \text{u} \quad \text{usuality constraint; } X \text{ isu } R \text{ means usually (} X \text{ is } R) \]

\[ r: \text{g} \quad \text{group constraint; } X \text{ isg } R \text{ means that } R \text{ constrains the attribute-values of the group} \]
PRIMARY GENERALIZED CONSTRAINTS

- Possibilistic: X is R
- Probabilistic: X isp R
- Veristic: X isv R

- Primary constraints are formalizations of three basic perceptions: (a) perception of possibility; (b) perception of likelihood; and (c) perception of truth

- In this perspective, probability may be viewed as an attribute of perception of likelihood
STANDARD CONSTRAINTS

- Bivalent possibilistic: $X \in C$ (crisp set)
- Bivalent veristic: $\text{Ver}(p)$ is true or false
- Probabilistic: $X$ is $p R$
- Standard constraints are instances of generalized constraints which underlie methods based on bivalent logic and probability theory
GENERALIZED CONSTRAINT LANGUAGE (GCL)

- GCL is an abstract language
- GCL is generated by combination, qualification, propagation and counterpropagation of generalized constraints
- Examples of elements of GCL
  - X/Age(Monika) is R/young (annotated element)
  - (X isp R) and (X,Y) is S)
  - (X isr R) is unlikely) and (X iss S) is likely
  - If X is A then Y is B
- The language of fuzzy if-then rules is a sublanguage of GCL

- Deduction = generalized constraint propagation and counterpropagation
CLARIFICATION

LANGUAGE VS. LANGUAGE SYSTEM

- **Language** = (description system)
- **Description system** = (syntax, semantics)
- **Language system** = (description system, computation/deduction system)
- **GCL is a language system**
THE CONCEPT OF GENERALIZED CONSTRAINT AS A BASIS FOR PRECISIATION OF MEANING

- Meaning postulate

Equivalently, mm-precisiation of \( p \) may be realized through translation of \( p \) into GCL.

Equivalently, mm-precisiation of \( p \) may be realized through translation of \( p \) into GCL.
EXAMPLES: POSSIBILISTIC

annotation

- Lily is young $\rightarrow$ Age (Lily) is young

- most Swedes are tall $\rightarrow$ Count (tall.Swedes/Swedes) is most
EXAMPLES: PROBABILISTIC

- $X$ is a normally distributed random variable with mean $m$ and variance $\sigma^2$  
  $X$ isp $N(m, \sigma^2)$

- $X$ is a random variable taking the values $u_1, u_2, u_3$ with probabilities $p_1, p_2$ and $p_3$, respectively  
  $X$ isp $(p_1\,u_1+p_2\,u_2+p_3\,u_3)$
COMPUTATION WITH INFORMATION DESCRIBED IN NATURAL LANGUAGE

- Representing the meaning of a proposition as a generalized constraint reduces the problem of computation with information described in natural language to the problem of computation with generalized constraints. In large measure, computation with generalized constraints involves the use of rules which govern propagation and counterpropagation of generalized constraints. Among such rules, the principal rule is the deduction principle (Zadeh 1965, 1975).
EXTENSION PRINCIPLE (POSSIBILISTIC)

- X is a variable which takes values in U, and f is a function from U to V. The point of departure is a possibilistic constraint on f(X) expressed as f(X) is A where A is a fuzzy relation in V which is defined by its membership function $\mu_A(v)$, $v \in V$.

- g is a function from U to W. The possibilistic constraint on f(X) induces a possibilistic constraint on g(X) which may be expressed as $g(X)$ is B where B is a fuzzy relation. The question is: What is B?
\[ \begin{align*}
&f(X) \text{ is } A \\
g(X) \text{ is } B
\end{align*} \]

\[ \mu_B(w) = \sup_u \mu_A(f(u)) \]

\[ w = g(u) \]

subject to

\[ \mu_A \text{ and } \mu_B \text{ are the membership functions of } A \text{ and } B, \text{ respectively.} \]
STRUCTURE OF THE EXTENSION PRINCIPLE

\[ u \leftarrow f^{-1}(A) \leftarrow \mu_A(f(u)) \]

\[ f(u) \leftarrow f^{-1}(A) \]

\[ g(f^{-1}(A)) \leftarrow g(f^{-1}(A)) \]

\[ f^{-1} \leftarrow f \]

counterpropagation

propagation
IMPRECISE PROBABILITY DISTRIBUTIONS
GRANULAR VS. GRANULE-VALUED DISTRIBUTIONS

\[ g(u): \text{probability density of } X \]

Possibility distribution of probability distributions

Probability distribution of possibility distributions
GRANULAR DISTRIBUTION
(PERCEPTION-BASED PROBABILITY DISTRIBUTION)

$X$ is a real-valued random variable

$P(X) = P_{i(1)}A_1 + P_{i(2)}A_2 + P_{i(3)}A_3$

$\text{Prob } \{X \text{ is } A_i\} \text{ is } P_{j(i)}$

$P(X) = \text{low\small + high\medium + low\large}$

BMD: $P(X) = P_{i(1)}A_1 + P_{i(2)}A_2 + P_{i(3)}A_3$

Prob $\{X$ is $A_i\}$ is $P_{j(i)}$

$P(X) = \text{low\small + high\medium + low\large}$
INTERPOLATION OF GRANULAR DISTRIBUTION

$p_1, p_2, ..., p_n$ are granular values of $p_i$, $i=1, ..., n$

$(P_i, A_i)$, $i=1, ..., n$ are given

$A$ is given

$(P, A)$

$g(u)$: probability density of $X$
INTERPOLATION MODULE AND PROBABILITY MODULE

\[ \text{Prob} \{X \text{ is } A_i\} = P_i \quad , \quad i = 1, \ldots, n \]

\[ \text{Prob} \{X \text{ is } A\} = Q \]

\[ \mu_Q(v) = \sup_g \left( \mu_{P_1} \left( \int_{U} \mu_{A_1}(u)g(u)du \right) \right) \wedge \cdots \wedge \]

\[ \mu_{P_n} \int_{U} \mu_{P_n} \left( \int_{U} \mu_{A_n}(u)g(u)du \right) \]

subject to

\[ U = \int_{U} \mu_A(u)g(u)du \]
EXAMPLE

- Probably it will take about two hours to get from San Francisco to Monterey, and it will probably take about five hours to get from Monterey to Los Angeles. What is the probability of getting to Los Angeles in less than about seven hours?

\[
BMD: (\text{probably, } *2) + (\text{probably, } *5) \\
\uparrow \uparrow \\
X \quad Y
\]

\[
Z = X + Y
\]

\[
\mathbb{p}_Z(w) = \int p_X(u)p_Y(w-u)du
\]
CONTINUED

query: $\int p_Z(w) \mu_{\leq \circlearrowleft 7}(w) \, dw$ is $\mathbb{A}$

$qri:$

$\prod p_X = \mu_{\text{probably}} (\int \mu_{\ast 2}(u) p_X(u) \, du)$

$\prod p_Y = \mu_{\text{probably}} (\int \mu_{\ast 5}(v) p_Y(v) \, dv)$

$\mu_{\mathbb{A}}(t) = \sup_{p_X, p_Y} (\prod_X \land \prod_Y)$

subject to:

$t = \int p_X(w) \mu_{\leq \circlearrowleft 7}(w) \, dw$
SUMMATION

- In the real world, imprecise probabilities are the norm rather than exception. In applications of probability theory it is a common practice to ignore imprecision of probabilities and treat them as if they were precise. The problem with this practice is that it leads to results whose validity is opened to question.
In the mainstream literature on imprecise probabilities the accent is on elicitation rather than on computation. The approach described in this lecture is a radical departure from the mainstream. The key assumptions are: (a) imprecise probabilities are a part of an environment of imprecise events, imprecise relations and imprecise constraints; and (b) imprecise probabilities and probability distributions are described in a natural language.
Computation with imprecise probabilities reduces to computation with information which is described in natural language, using the formalism of Computing with Words (CW) and, more generally, NL Computation. The centerpiece of these formalisms is the concept of a generalized constraint. The concept of a generalized constraint plays a key role in precisiation of meaning.
RELATED PAPERS BY L.A.Z IN REVERSE CHRONOLOGICAL ORDER


CONTINUED

- From computing with numbers to computing with words -- from manipulation of measurements to manipulation of perceptions, IEEE Transactions on Circuits and Systems 45, 105-119, 1999.


CONTINUED


- Fuzzy probabilities and their role in decision analysis, Proc. MIT/ONR Workshop on C\3\d, MIT, Cambridge, MA., 1981.

- Fuzzy sets vs. probability, (correspondence item), Proc. IEEE 68, 421, 1980.

**PRECISIATION AND COMPUTATION/DEDUCTION—EXAMPLE**

- **p**: most Swedes are tall
  - \( p^* : \sum \text{Count(tall.Swedes/Swedes)} \) is most
- **q**: How many are short?
  - further precisiation

- \( X(h) \): height density function (not known)
- \( X(h)du \): fraction of Swedes whose height is in \([h, h+du]\), \( a \leq h \leq b \)

\[
\int_a^b X(h)du = 1
\]
• fraction of tall Swedes: \( \int_a^b X(h) \mu_{\text{tall}}(h) \, dh \)

• constraint on \( X(h) \)

\[ \int_a^b X(h) \mu_{\text{tall}}(h) \, dh \] is most granular value

\[ \pi(X) = \mu_{\text{most}}(\int_a^b X(h) \mu_{\text{tall}}(h) \, dh) \]
CONTINUED

deduction:

\[ \int_{a}^{b} X(h) \mu_{\text{tall}}(h) \, dh \]

is most \( \leftarrow \) given

\[ \int_{a}^{b} X(h) \mu_{\text{short}}(h) \, dh \]

is \( ? \) \( Q \leftarrow \) needed

solution:

\[ \mu_{Q}(v) = \sup_{X} (\mu_{\text{most}}(\int_{a}^{b} X(h) \mu_{\text{tall}}(h) \, dh)) \]

subject to

\[ v = \int_{a}^{b} X(h) \mu_{\text{short}}(h) \, dh \]

\[ \int_{a}^{b} X(h) \, dh = 1 \]