CS 70 Spring 2005 — Discrete Mathematics for CS — Clancy/Wagner

Final

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your class account name: cs70- _____________

Name of the person sitting to your left: ___________________________

Name of the person sitting to your right: ___________________________

You may consult any books, notes, or other paper-based inanimate objects available to you. Calculators and computers are not permitted. Please write your answers in the spaces provided in the test; in particular, we will not grade anything on the back of an exam page unless we are clearly told on the front of the page to look there. If you use a result from the lecture notes, the homework solutions, or the Rosen textbook, please indicate clearly where in the course material your result appears.

You have 180 minutes. There are 8 questions, of varying credit (60 points total). We believe that the harder questions appear at the end of the exam. The questions are of varying difficulty, so avoid spending too long on any one question.

Do not turn this page until your instructor tells you to do so.
Then, print your login name on the subsequent pages of the exam.

Problem 1		Problem 5	
Problem 2		Problem 6	
Problem 3		Problem 7	
Problem 4		Problem 8	
		Total	

Problem 1. (8 points)

Let $a(n)$ denote the number of ternary strings of length n with no pair of adjacent 2's. Example: "2011" is a ternary string with no pair of adjacent 2's.

(a) What are $a(1)$ and $a(2)$?

(b) Find a recurrence relation for $a(n)$.

(c) Prove that $a(n) \geq 2^{n+1}$ for all $n \geq 2$.

Problem 2. (4 points)

Solve the following system of equations modulo 7 for x and y. Show your work.

$$y \equiv 5x - 3 \pmod{7}$$
$$y \equiv 3x + 2 \pmod{7}$$

Problem 3. (4 points)

Suppose there are two males and two females to be paired. Prove that in every stable pairing, at least two individuals get their first-choice mate.

Problem 4. (10 points)

Shuffle a standard 52-card deck, and deal out a hand of 13 cards. You get 4 points for each Ace in the hand, 3 points for each King, 2 for each Queen, 1 for each Jack, and nothing for the other cards. Let the random variable X denote the total number of points in your hand.

(a) Calculate $\Pr[X = 0]$. You may leave your answer as an unevaluated expression. Show your work.

(b) Calculate $\mathbf{E}[X]$. This time we want a number; again, show your work.

(c) Suppose we tell you that your hand has 6 spades, 2 hearts, 3 diamonds, and 2 clubs. Now consider the expected value of X, conditioned on this fact. Does this number increase, decrease, or stay the same, compared to your answer in part (b)? Briefly justify your answer.

Problem 5. (9 points)

We have a room full of m people. Assume that each person's birth date is uniformly and independently distributed among the set of 365 possibilities. Let the random variable X denote the number of different birth dates among all the people in the room.

Example: Alice's birth date is April 5, Bob's is June 23, Carol's is April 5. There are two different birth dates, so $X = 2$ in this case.

(a) Calculate $\mathbf{E}[X]$. Your answer should be a simple function of m.

(b) Determine how many ways there are to assign a birth date to each of the *m* people in the room, so that we end up with exactly 3 different birthdays among them all. Show your work.

Problem 6. (7 points)

Eric is feeling generous today. Earlier, he secretly chose a number X uniformly at random from the set $\{1, 2, 3, \ldots, 100\}$ and put $\$X$ into one box and $\$2X$ into another (identical) box.

You randomly choose a box by flipping a fair coin. Eric is going to show you what is in the box you chose, and then you will have the option of either taking the box you chose or trading boxes.

(a) Suppose you see \$8 in the box you chose. Given this, what is the probability that the other box contains \$16?

(b) Suppose you see \$8 in the box you chose. What is the expected value of your winnings, if you choose to stick with the box you chose?

(c) Suppose you see \$8 in the box you chose. What is the expected value of your winnings, if you choose to swap boxes? Should you stick or switch?

(d) Suppose instead that you see \$7 in your chosen box. Should you stick or switch? Briefly justify your answer.

Problem 7. (6 points)

Let F_k denote the kth Fibonacci number, defined by $F_0 = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$. Let $\textsc{ExtEuclid}(x, y)$ denote the result of running the extended Euclidean algorithm on inputs x and y.

What does $\textsc{ExtEuclid}(F_{k+1}, F_k)$ output? Show how you got your result.

Problem 8. (12 points)

Mike challenges David to think of a polynomial in one variable— $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$— whose coefficients are all nonnegative integers. Mike claims that he will ask for just two evaluations of the polynomial, and then he will tell David its coefficients. Here's how.

1. Mike asks David to evaluate $P(1)$. He calls that value r.
2. Mike asks David to evaluate $P(r+1)$. After receiving the answer and giving it some thought, he reads off the coefficients of P.

Here's an example. Mike asks for $P(1)$, and David says 9. Mike then asks for $P(10)$. David says 342. Mike correctly identifies P as $3x^2 + 4x + 2$.

(a) Demonstrate that knowing $r = P(1)$ and $y = P(r)$ does *not* provide you with enough information to uniquely determine P.

(b) Explain, in terms detailed enough for another CS 70 student to be able to implement your algorithm immediately, how to determine P efficiently.

(c) Give a big-Oh estimate that's as accurate as possible for the number of arithmetic operations on integers required to execute the algorithm you described in part (b). Justify your answer.