Due Thursday, April 7

Coverage: This assignment involves topics from the lectures of March 15, 17, 29, and 31, and from Rosen sections 4.4 and 5.1. The assignment is shorter than usual because of the April 5 exam.

Administrative reminders: We will accept only unformatted text files or PDF files for homework submission. Include your name, login name, section number, and partner list in your submission. Give the command `submit hw9` to submit your solution to this assignment.

Homework exercises:

1. **(50 pts.) Binomial coefficients**
This question will teach you about binomial coefficients. As was explained in class, suppose we want to count how many different ways there are to choose a subset of m objects from a set of n different objects. Assume that repetition is not allowed, and the order in which you choose the m objects is ignored. The answer, of course, is $\frac{n!}{m!(n-m)!}$. This is such an important value that we define the notation $\binom{n}{m}$ as short-hand for $\frac{n!}{m!(n-m)!}$. This is called the binomial coefficient "n choose m", and you will sometimes see it written $C(n,m)$ in other texts. As a matter of convention, if $m < 0$ or $m > n$, we define $\binom{n}{m}$ to be 0.

Also, this question will show you different ways to think about the same problem. Often in combinatorics there are two ways to approach a problem: by thinking combinatorially (in terms of counting objects), or by thinking algebraically (manipulating equations from their definitions as mathematical formulas). Both modes of thinking are worth having in your toolbox, and this question asks you to explore the interplay between them.

With that background, answer the following questions.

(a) Let S be a set of size 10. How many different subsets of size 4 does S have?

Binomial coefficients satisfy many useful identities. One of them is that

$$\binom{n}{m} = \binom{n}{n-m}.$$

A combinatorial "argument" might help give the intuition of why the latter should be true: the number of ways to select m objects to keep is the same as the number of ways to select $n-m$ objects to throw away (i.e., not to keep).

(b) Prove the above identity algebraically, straight from the definition of $\binom{n}{m}$.

Another useful identity is

$$\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}.$$

A combinatorial explanation is as follows. There are $\binom{n+1}{m+1}$ ways to choose a homework group of $m+1$ students out of a classroom of $n+1$ students. Consider any student in the room, let's say Alice. Every homework group can be classified in one of two ways: either it contains Alice, or it doesn't. There are $\binom{n}{m}$ homework groups in the former category (because the group contains Alice, as well as m students selected from the remaining population of the class), and $\binom{n}{m+1}$ students in the latter category (because one must select $m+1$ students from the class excluding Alice). Thus there are $\binom{n}{m} + \binom{n}{m+1}$ ways to choose a homework group. We've counted the same quantity in two different ways, so the two answers must be equal, and equating them gives the identity.

(c) Now give an algebraic proof of the previous identity, straight from the definition of $\binom{n}{m}$.

Another way to interpret binomial coefficients is in terms of walking the blocks of a city square. See Figure 1. A person starts at the topmost intersection (the origin) and walks down the streets; at each intersection, the person can walk either down-and-left or down-and-right. We can associate each intersection with the coordinates (n, ℓ), where n counts how many blocks you need to walk to get there from the origin and ℓ counts the number of leftward blocks you need to follow.

(d) Argue that the number of paths from the origin to the intersection labelled (n, ℓ) is exactly $\binom{n}{\ell}$.

A third useful identity is the following:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

(e) Justify this identity, with either a combinatorial or algebraic argument (you choose which one to use).

A fourth identity is the following:

$$\binom{m}{m} + \binom{m+1}{m} + \binom{m+2}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}.$$

(f) Justify this identity, with either a combinatorial or algebraic argument (you choose which one to use).

(g) Let $S_n = 1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + (n-2) \times (n-1) \times n$. Find a simple expression for S_n.

2. (10 pts.) Money bags

I have a bag containing either a \$1 or \$5 bill (with equal probability assigned to both possibilities). I then add a \$1 bill to the bag, so it now contains two bills. The bag is shaken, and you draw out a \$1 bill. If a second student draws the remaining bill from the bag, what is the chance that it is a \$1 bill?

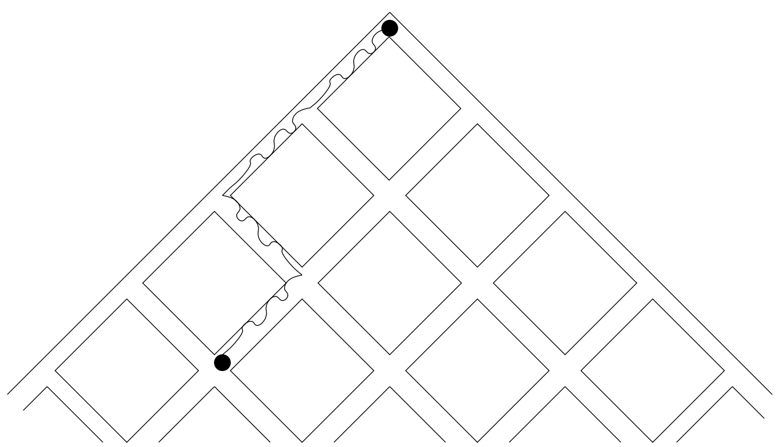

Figure 1: A "block-walking" interpretation of binomial coefficients. The example shows one path from the origin $(0,0)$ to the point $(4,3)$.