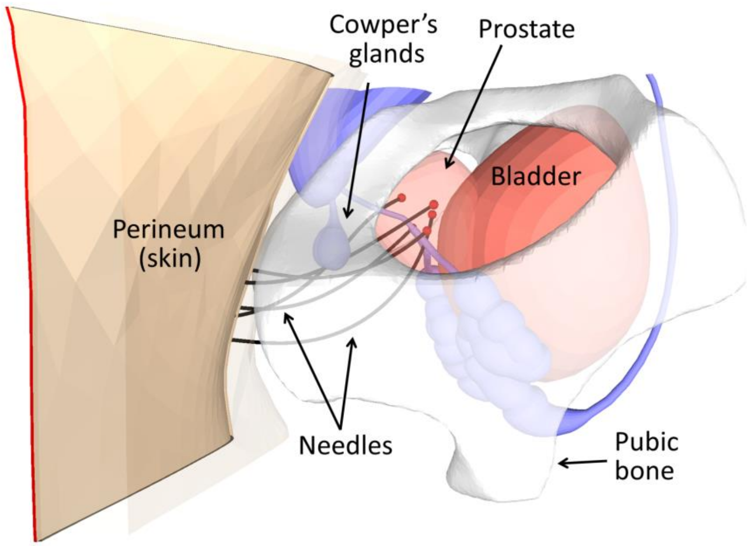
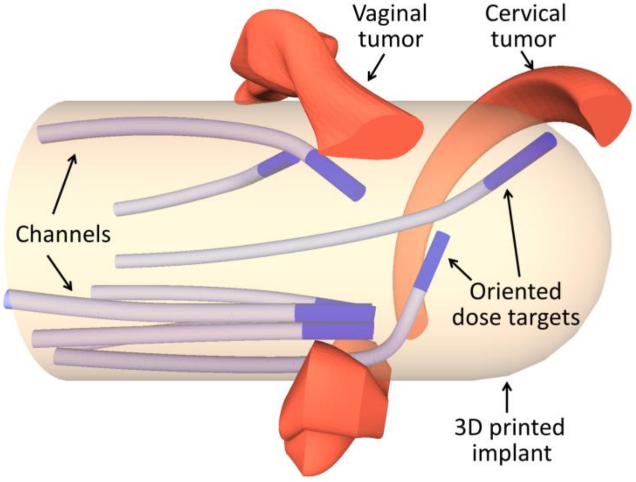




Optimization over Manifolds with applications to Robotic Needle Steering and Channel Layout Design

Sachin Patil

Guest Lecture: CS287 Advanced Robotics

Trajectory Optimization

$$\min_{\boldsymbol{\theta}_{1:T}} \sum_t \|\boldsymbol{\theta}_{t+1} - \boldsymbol{\theta}_t\|^2 + \text{ other costs}$$

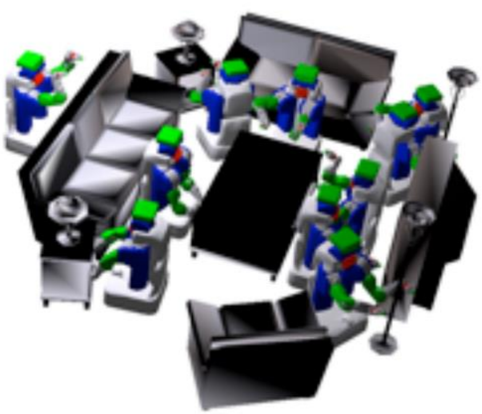
subject to

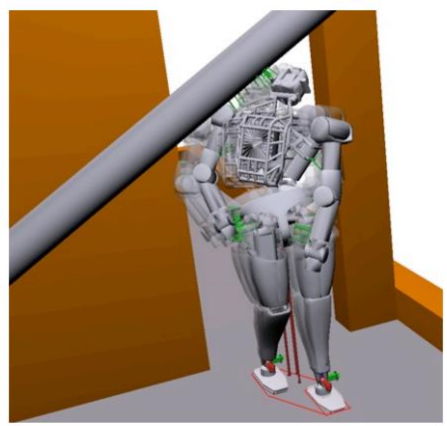
no collisions

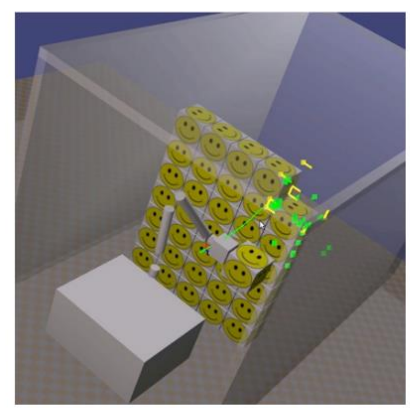
joint limits

other constraints

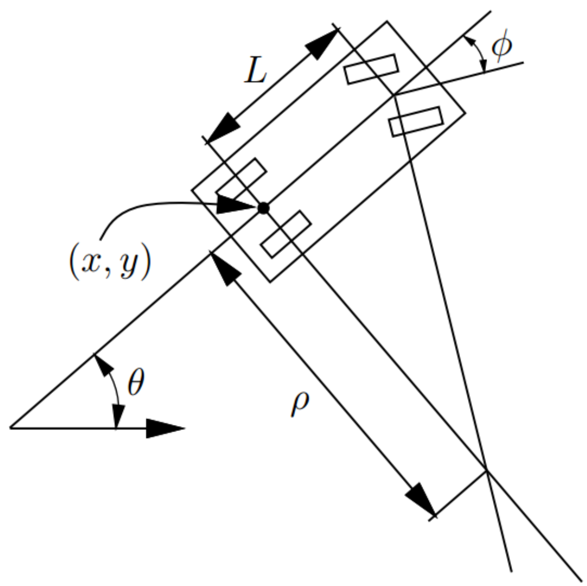
← Optimization over vector spaces $\mathbb{R}^n$

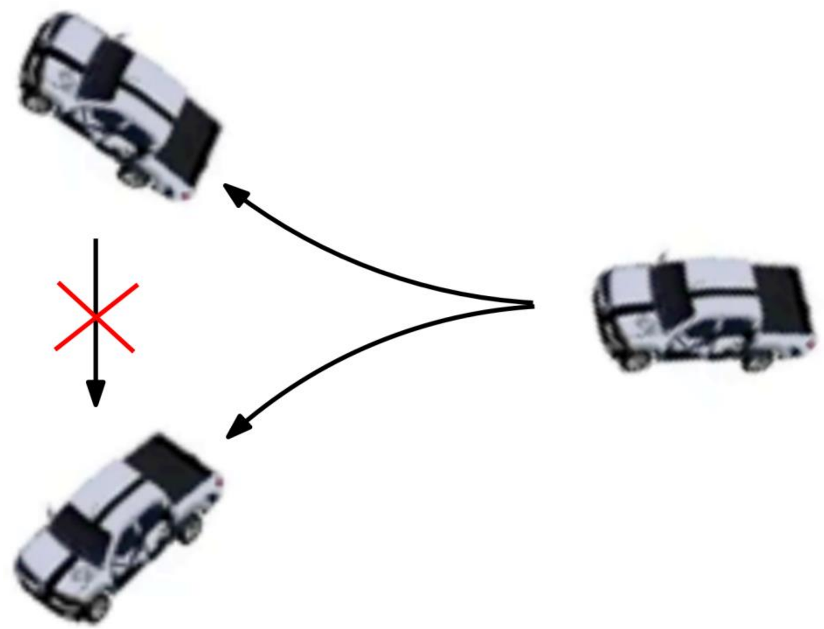






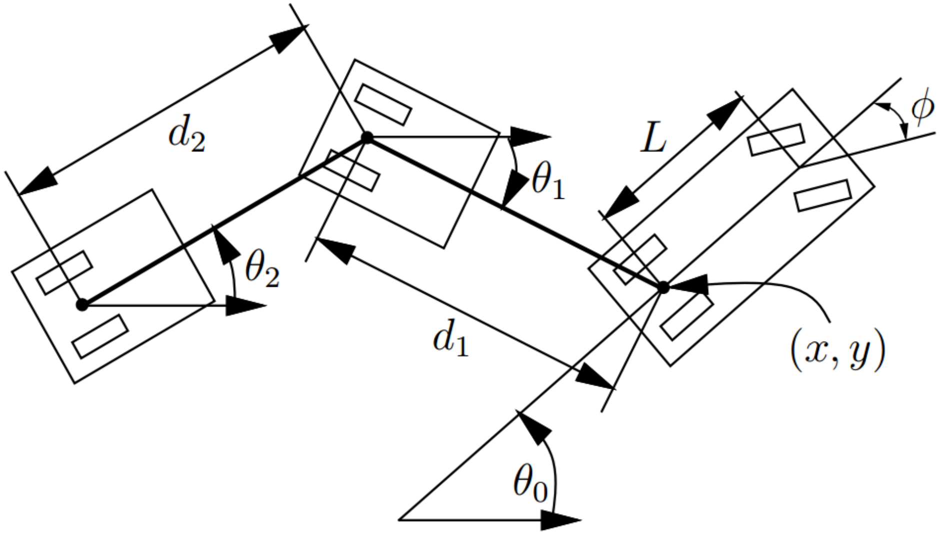
Not All State-Spaces are 'Nice'

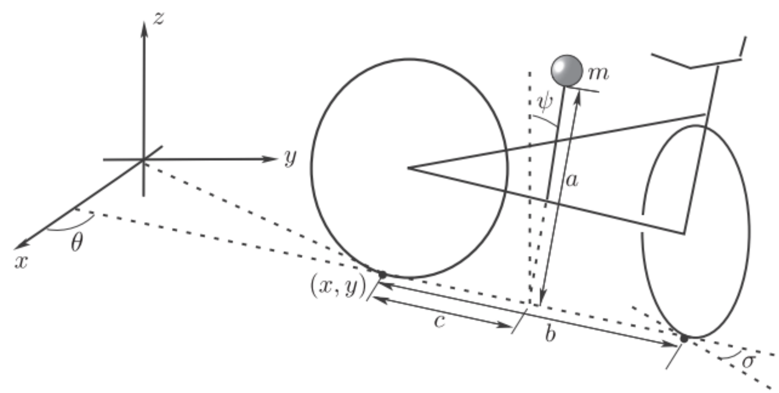




- **Nonholonomic** system cannot move in arbitrary directions in its state space
- For a simple car: Configuration space is in $\mathbb{R}^2 \times \mathbb{S}^1 : [x, y, \theta]$ (the SE(2) group)

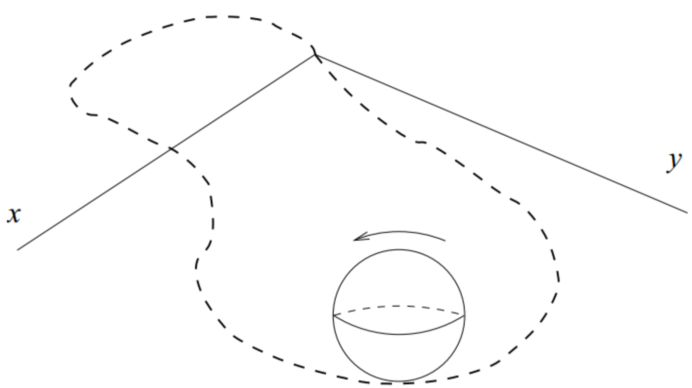
Nonholonomy Examples





Car pulling trailers: $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \ldots \mathbb{S}^1$

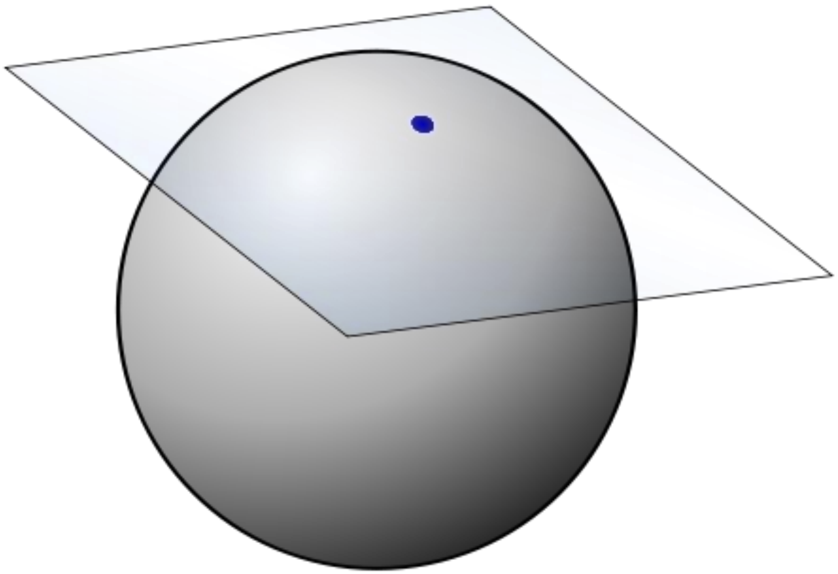
Bicycle: $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$



Rolling Ball: ?

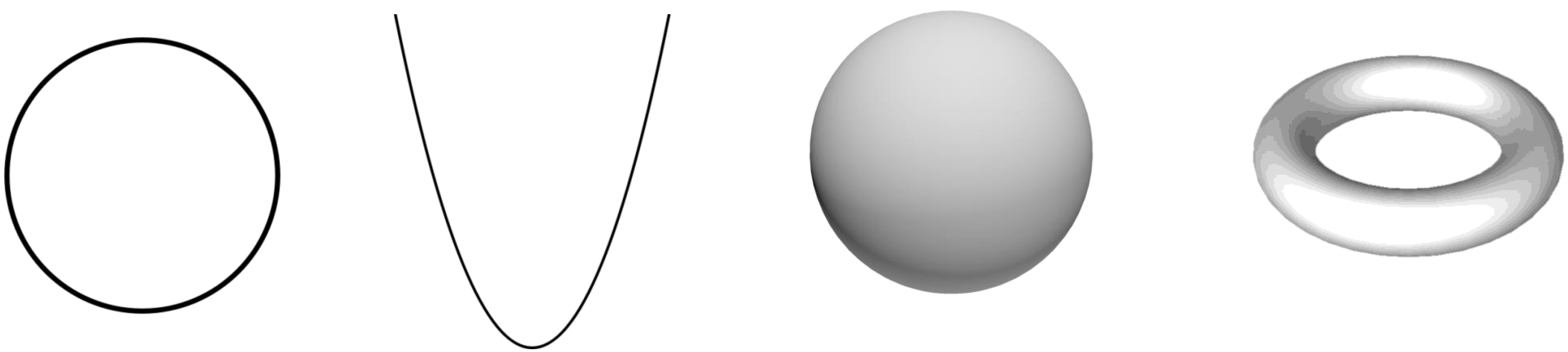
$\mathbb{R}^2 \times SO(3)$

C-Spaces as Manifolds

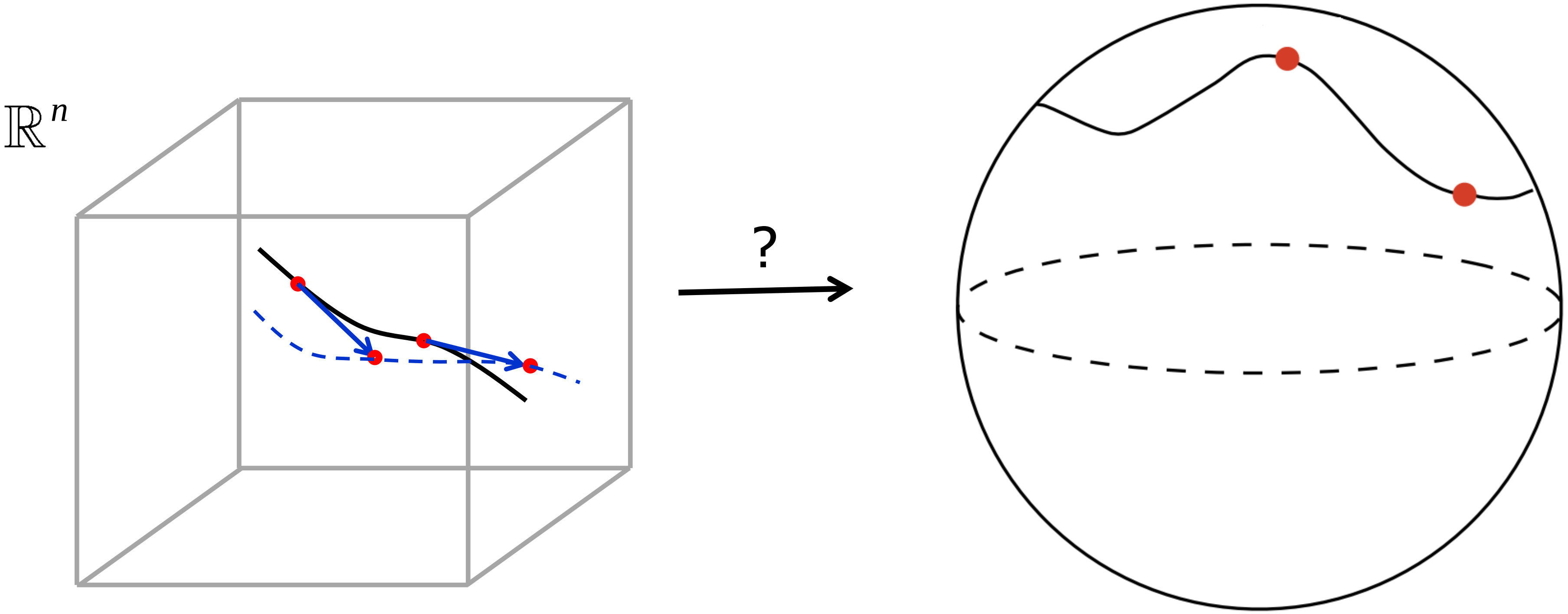


Manifold: Topological space that near each point resembles Euclidean space

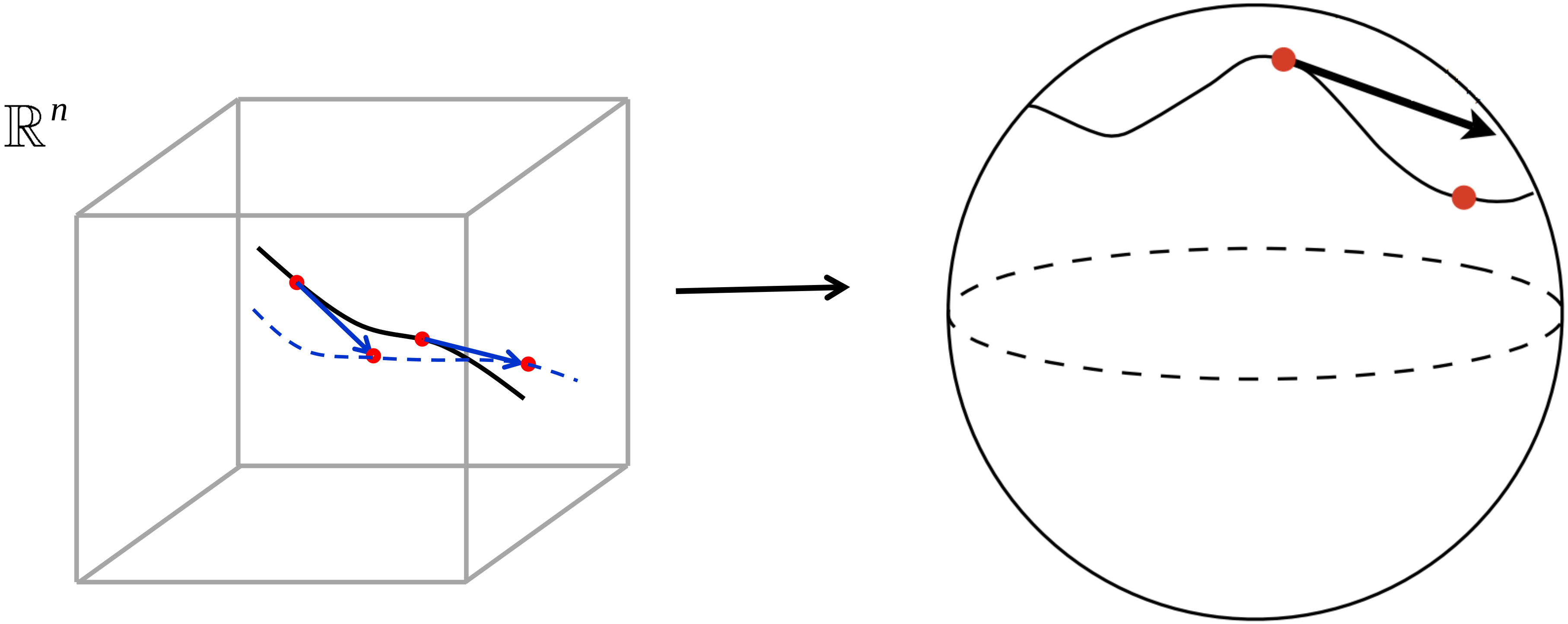
Other examples:



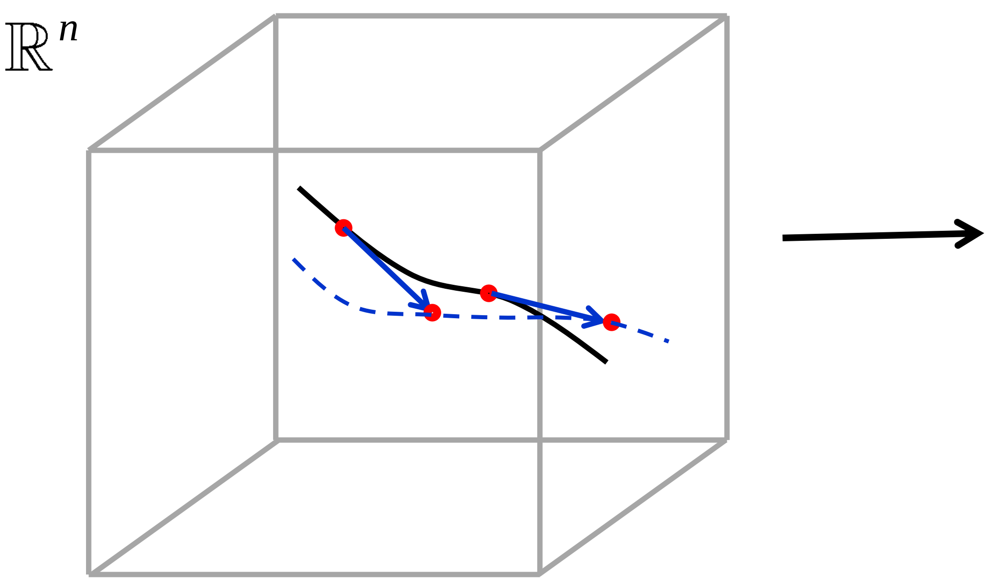
Optimization over Manifolds

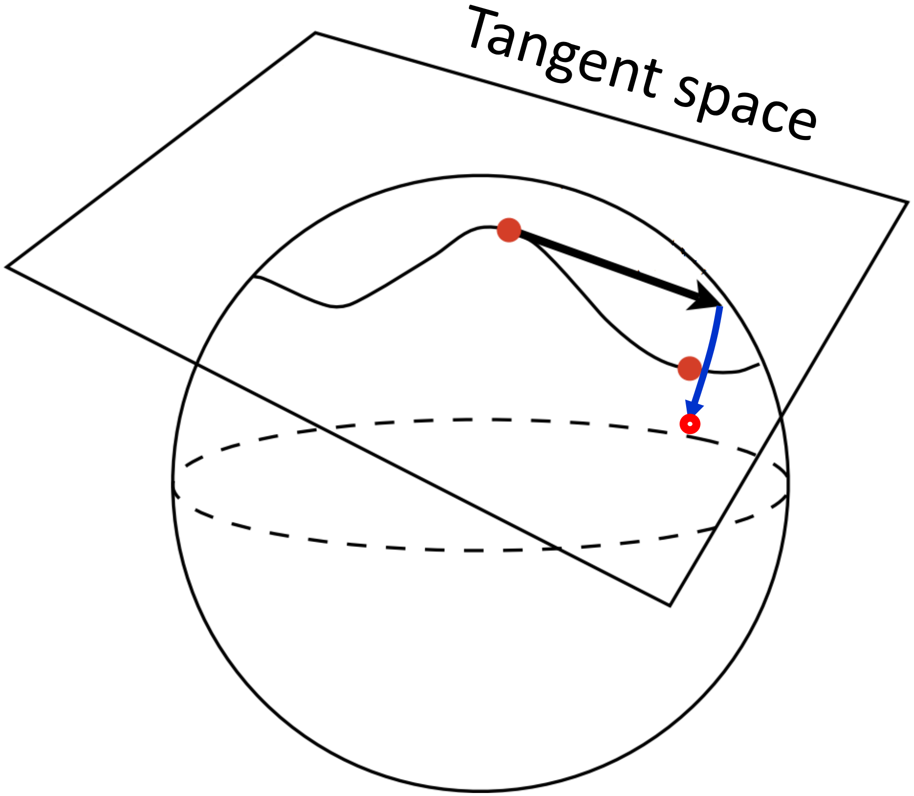


Optimization over Manifolds



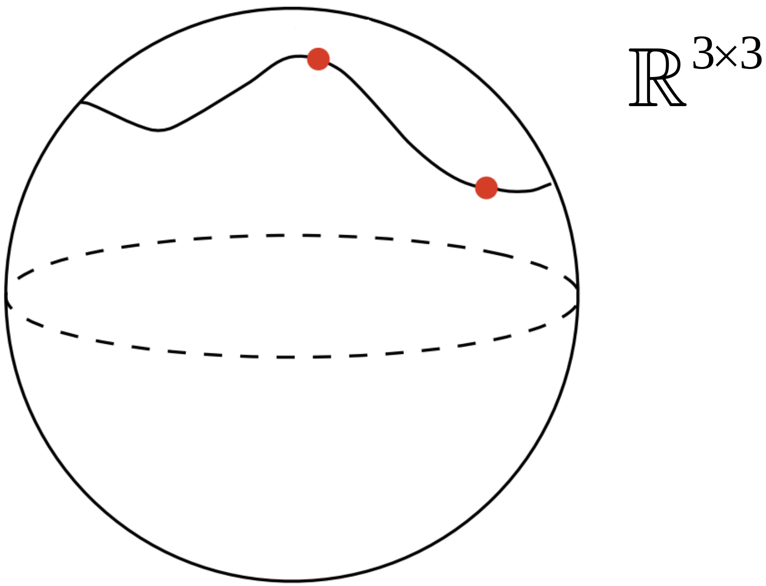
Optimization over Manifolds





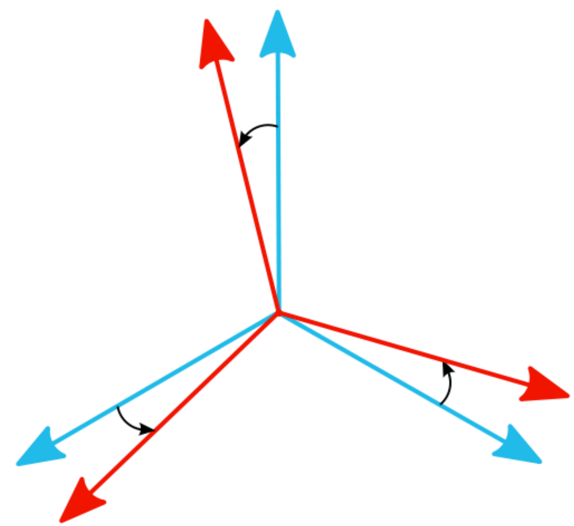
Define projection operator from tangent space to manifold

Case Study: Rotation Group (SO(3))



Rotation matrices:

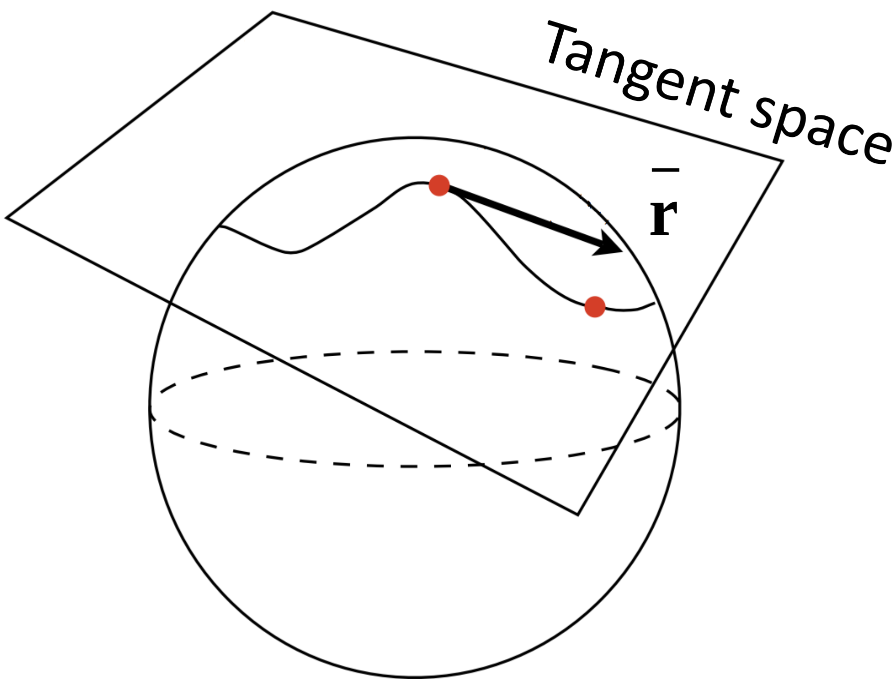
- Unique representation
- ‘Smooth’



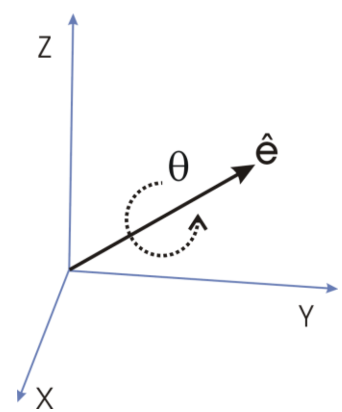
Optimization over SO(3) arises in robotics, graphics, vision etc.

Parameterization: Incremental Rotations

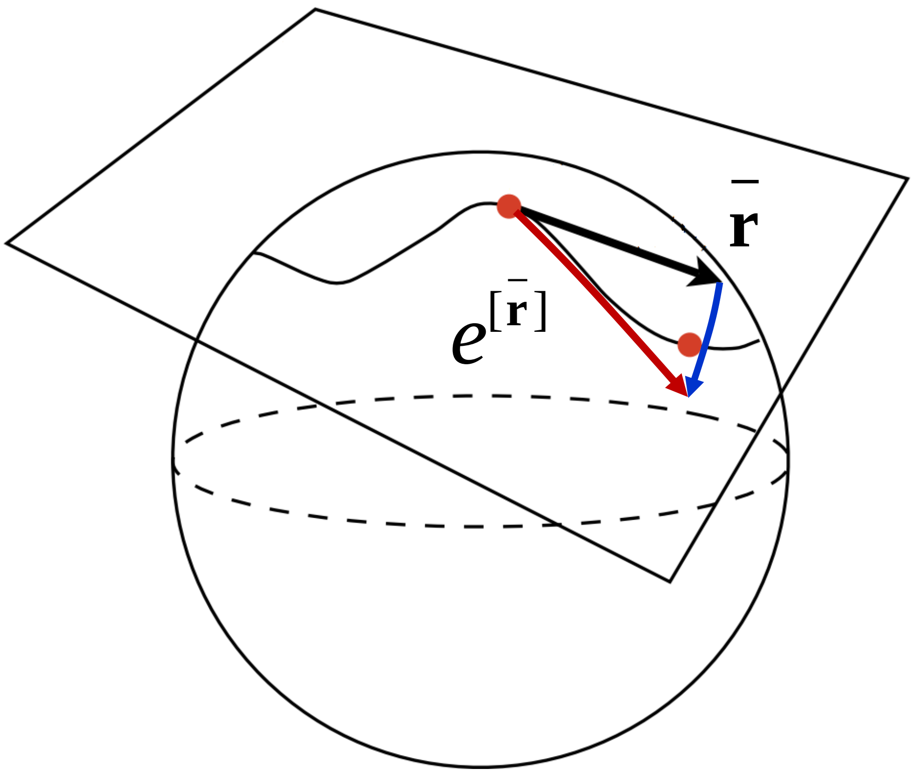
- Why not directly optimize over rotation matrix entries?
  - Over-constrained (orthonormality)
  - Larger number of optimization variables
- Define local parameterization in terms of incremental rotation



$\bar{\mathbf{r}}$: Incremental rotation to reference rotation defined in terms of axis-angle



Projection Operator



$e^{[\bar{\mathbf{r}}]}$: Point on SO(3) that can be reached by traveling along the geodesic in direction $\bar{\mathbf{r}}$

where $[\bar{\mathbf{r}}] = \begin{bmatrix} 0 & -\bar{\mathbf{r}}_z & \bar{\mathbf{r}}_y \\ \bar{\mathbf{r}}_z & 0 & -\bar{\mathbf{r}}_x \\ -\bar{\mathbf{r}}_y & \bar{\mathbf{r}}_x & 0 \end{bmatrix}$

and $e^X = \sum_{k=0}^{\infty} \frac{1}{k} X^k$ is the

matrix exponential operator

Optimization Procedure

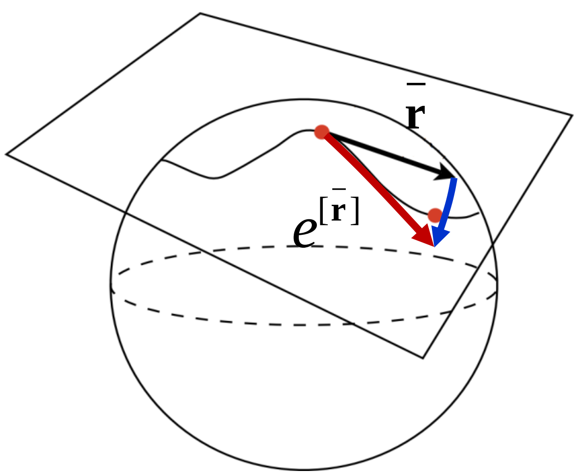
1) Seed trajectory: $\mathcal{R}^{i} = [\hat{R}_1^i, \ldots, \hat{R}_n^i]$

2) $\min$ Objective subject to: Constraints

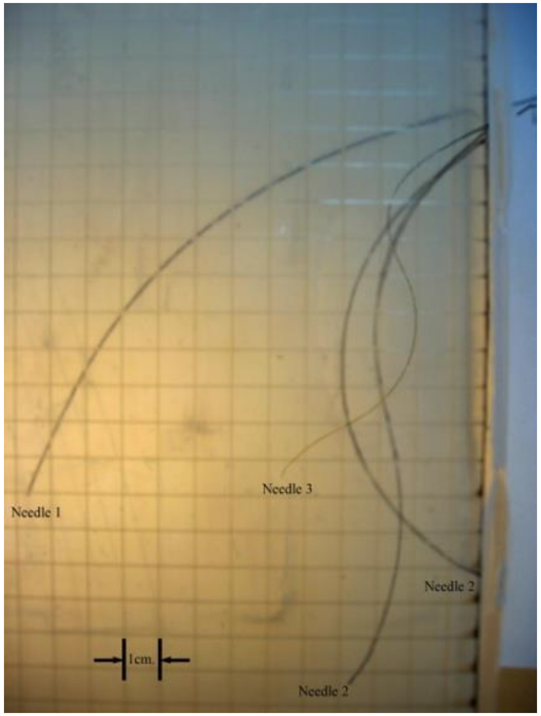
   $\overline{\mathcal{R}}^{i} = [\bar{\mathbf{r}}_1^{i}, \ldots, \bar{\mathbf{r}}_n^{i}]$

3) Compute new trajectory: $\mathcal{R}^{i+1} = [\hat{R}_1^i \cdot e^{[\bar{\mathbf{r}}_1^{i}]}, \ldots, \hat{R}_n^i \cdot e^{[\bar{\mathbf{r}}_n^{i}]}]$

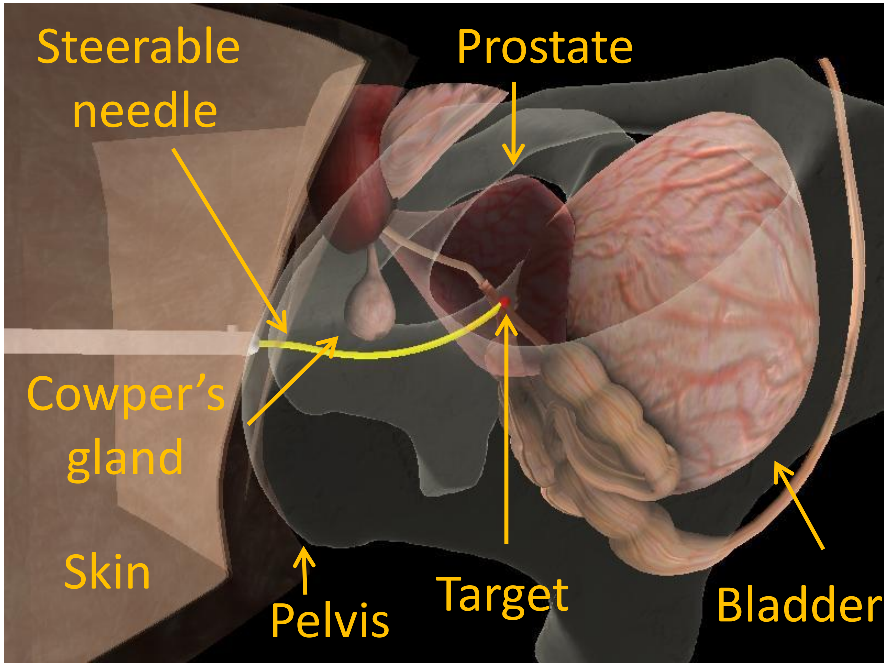
4) Reset increments: $\overline{\mathcal{R}}^{i+1} = [\mathbf{0}, \ldots, \mathbf{0}]$



Steerable Needle



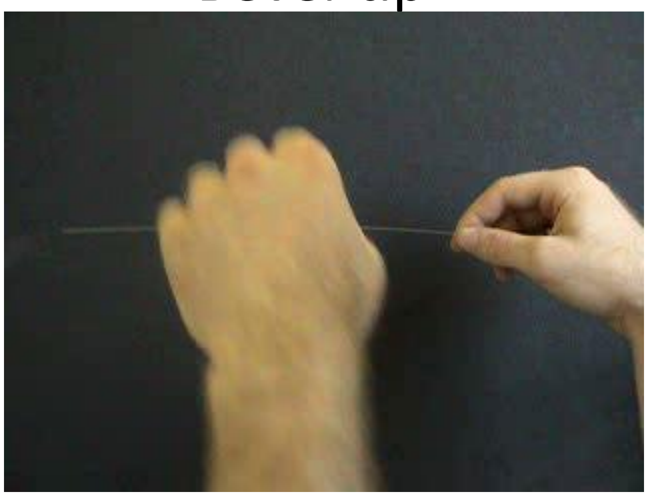
Steerable needles inside phantom tissue



Steerable needles navigate around sensitive structures (simulated)

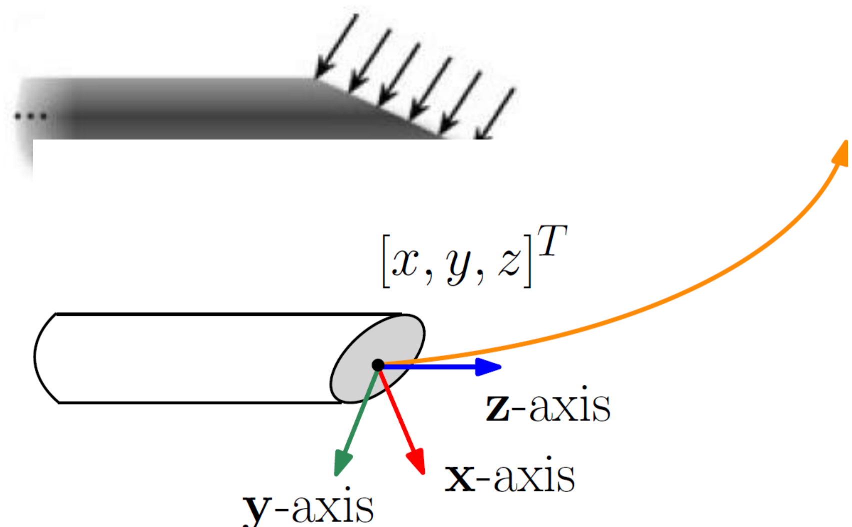
Steerable Needle

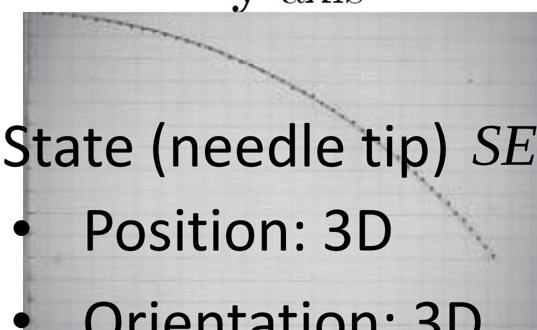
Bevel-tip



Highly flexible

[Webster, Okamura, Cowan, Chirikjian, Goldberg, Alterovitz
United States Patent 7,822,458. 2010]





State (needle tip) $SE(3) : \mathbb{R}^3 \times SO(3)$

- Position: 3D
- Orientation: 3D

Follows constant curvature paths

Steerable Needle: Opt Formulation

$$\min_{\bar{\mathcal{X}},\mathcal{U}} \ \alpha_\Delta \mathrm{Cost}_\Delta + \alpha_\phi \mathrm{Cost}_\phi + \alpha_\mathcal{O} \mathrm{Cost}_\mathcal{O},$$

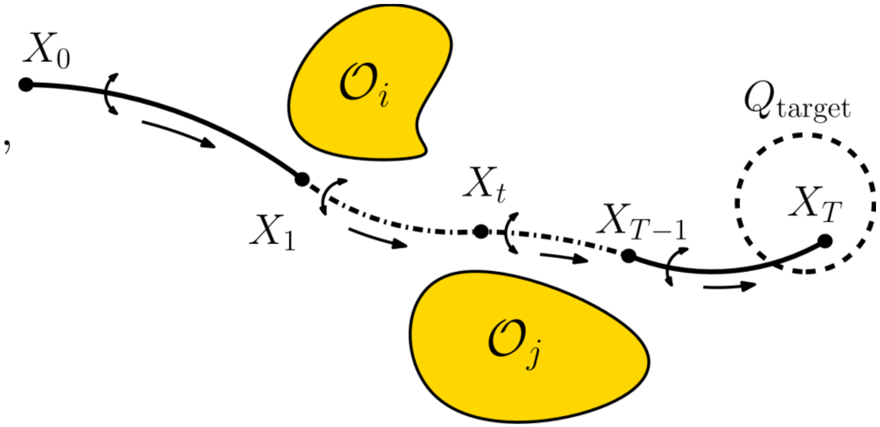
$$\text{s.t.} \quad \log((X_t \cdot \exp(\mathbf{w}_t^\wedge) \cdot \exp(\mathbf{v}_t^\wedge))^{-1} \cdot X_{t+1})^\vee = \mathbf{0}_6,$$

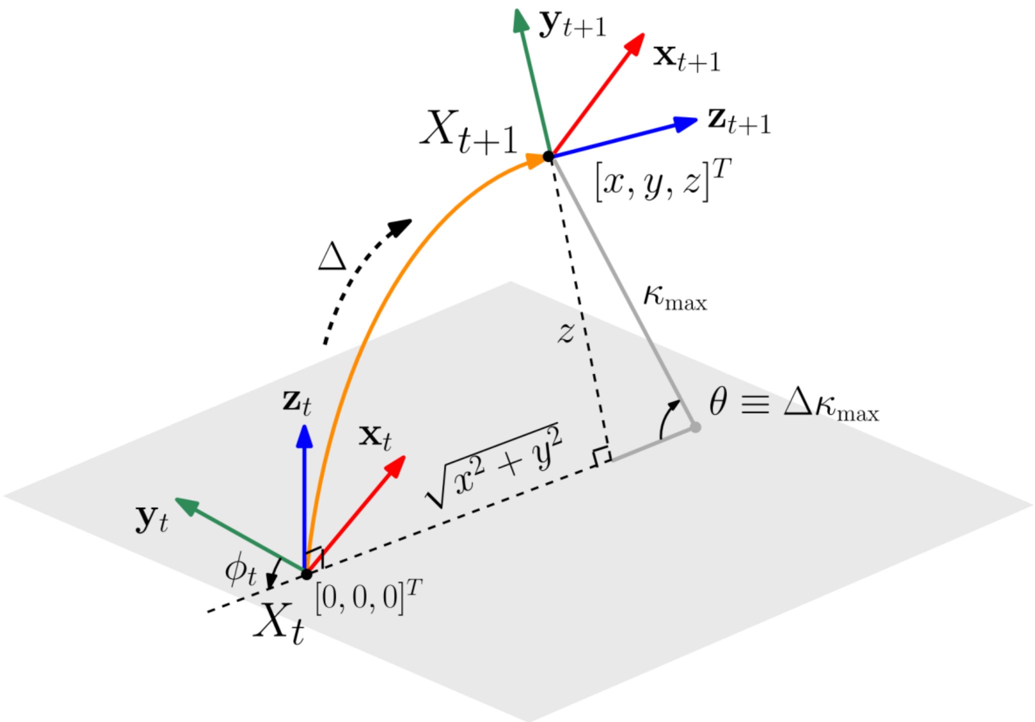
$$\mathrm{sd}(X_t, X_{t+1}, \mathcal{O}_i) \geq d_{\mathrm{safe}} + d_{\mathrm{arc}},$$

$$X_0 \in Q_{\mathrm{entry}}, \ \ X_T \in Q_{\mathrm{target}},$$

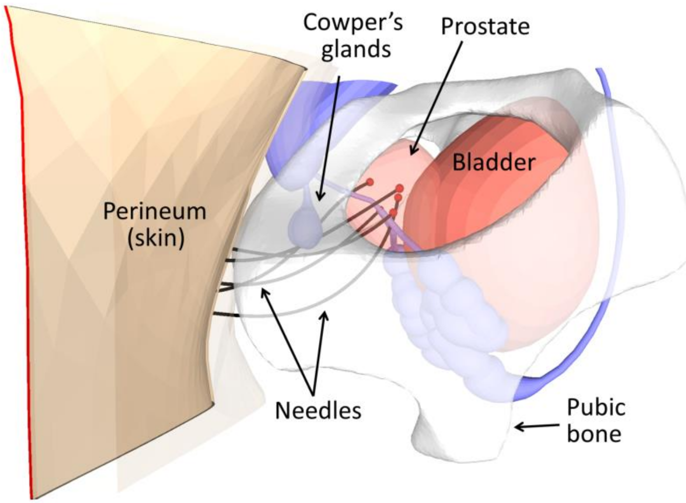
$$-\pi \leq \phi_t \leq \pi,$$

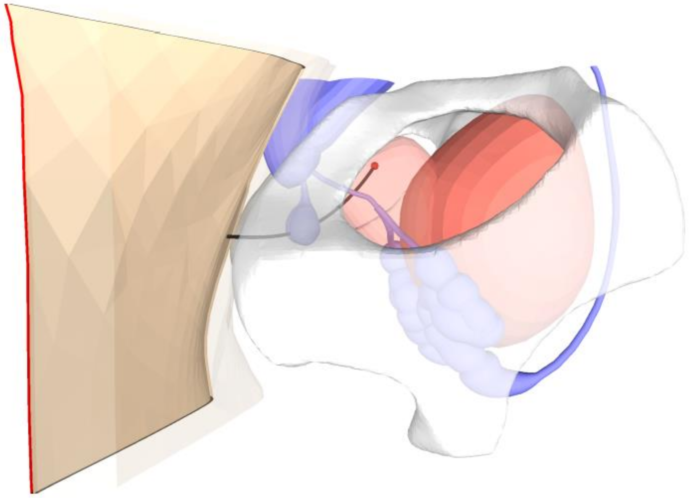
$$\kappa_t = \kappa_{\max}$$



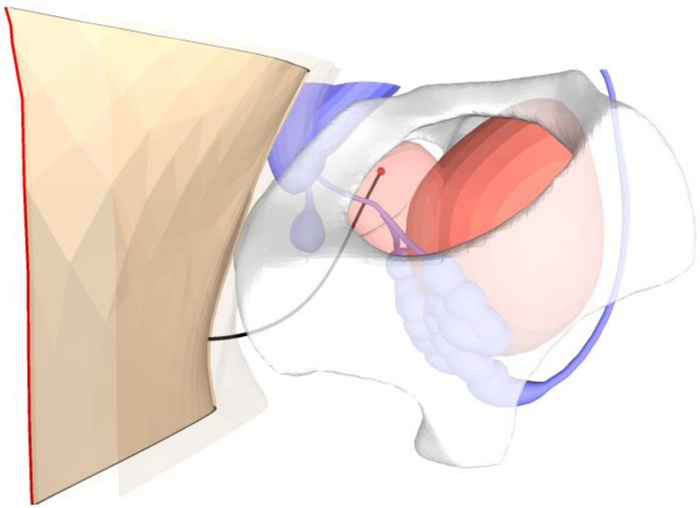


Steerable Needle Plans





(a) Smaller clearance from obstacles (Cowper's glands) with $\alpha_{\mathcal{O}} = 1$.



(b) Larger clearance from obstacles with $\alpha_{\mathcal{O}} = 10$.

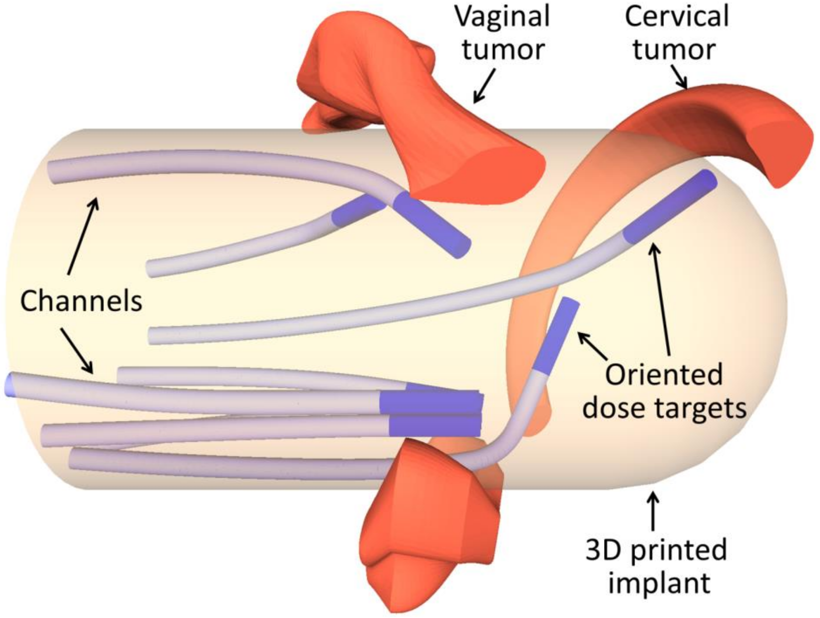
Results

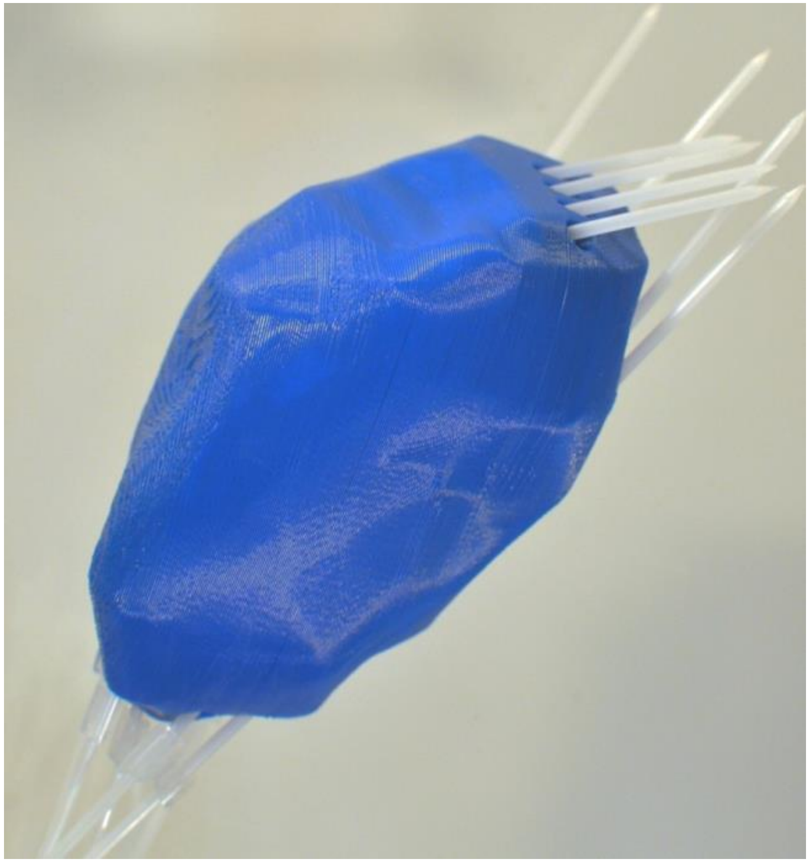
	RRT	collocation $\alpha_{\mathcal{O}} = 1$	shooting $\alpha_{\mathcal{O}} = 1$	collocation $\alpha_{\mathcal{O}} = 10$	shooting $\alpha_{\mathcal{O}} = 10$
solved%	67.3%	76.0%	80.3%	79.0%	89.5%
time (s)	9.8 ± 8.1	1.8 ± 1.2	1.6 ± 1.7	1.9 ± 1.3	1.8 ± 1.7
path length	11.1 ± 1.5	11.3 ± 1.4	11.6 ± 1.7	11.9 ± 1.7	13.1 ± 2.3
twist cost	34.9 ± 10.0	1.4 ± 1.4	1.0 ± 1.0	1.6 ± 1.6	1.0 ± 1.0
clearance	0.5 ± 0.4	0.7 ± 0.5	0.5 ± 0.3	1.3 ± 0.4	1.2 ± 0.5

Performance of our approach on the single needle planning case.

Why is minimizing twist important?

Channel Layout (Brachytherapy Implants)





Channel Layout: Opt Formulation

$$\min_{\bar{\mathcal{X}},\mathcal{U}} \ \alpha_\Delta \mathrm{Cost}_\Delta + \alpha_\phi \mathrm{Cost}_\phi + \alpha_{\mathcal{O}} \mathrm{Cost}_{\mathcal{O}},$$

$$\text{s.t.} \quad \log((X_t \cdot \exp(\mathbf{w}_t^\wedge) \cdot \exp(\mathbf{v}_t^\wedge))^{-1} \cdot X_{t+1})^\vee = \mathbf{0}_6,$$

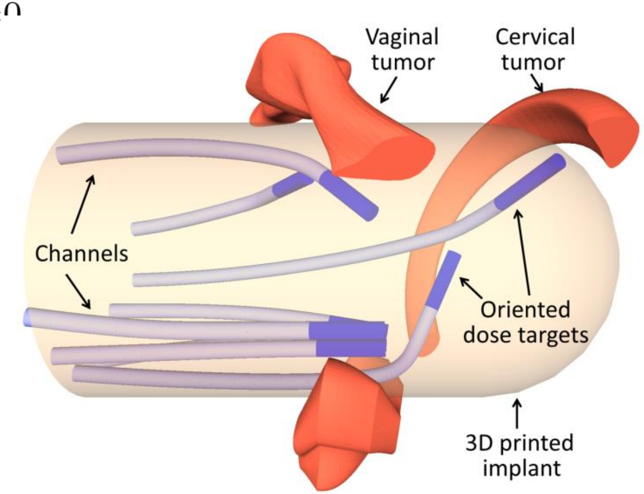
$$\mathrm{sd}(X_t, X_{t+1}, \mathcal{O}_i) \geq d_{\text{safe}} + d_{\text{arc}},$$

$$X_0 \in Q_{\text{entry}}, \ X_T \in Q_{\text{target}},$$

$$-\pi \leq \phi_t \leq \pi,$$

$$0 \leq \kappa_t \leq \kappa_{\max},$$

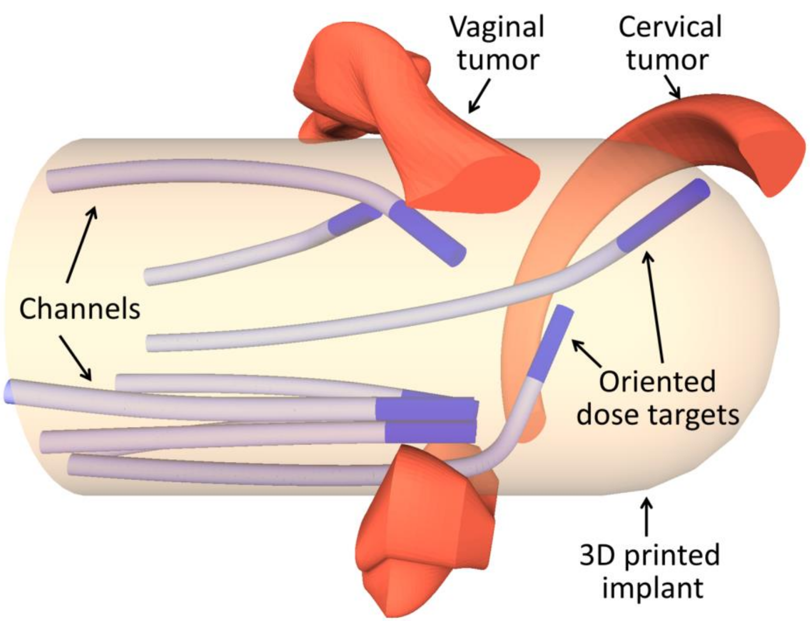
$$\Delta \sum_{t=0}^{T-1} \kappa_t \leq c_{\max} \quad \text{for channel planning},$$



Results

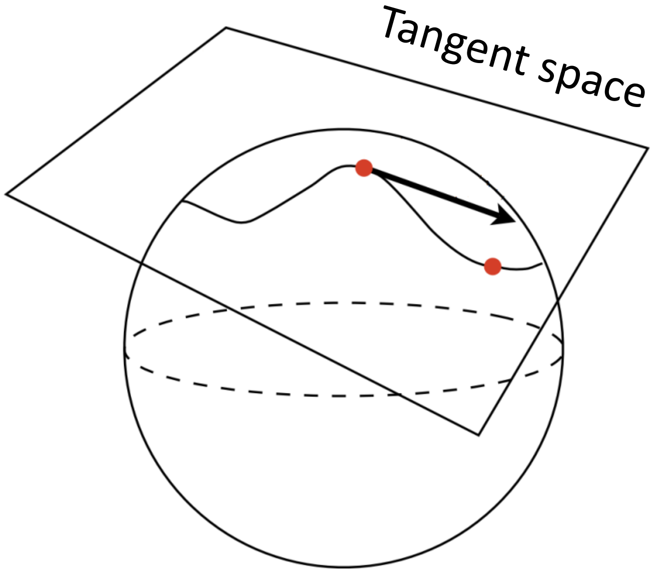
	RRT	backward shooting
solved%	74.0%	98.0%
time (s)	30.8 ± 17.9	27.7 ± 9.8
path length	41.3 ± 0.3	38.9 ± 0.1
twist cost	65.5 ± 8.4	4.1 ± 1.1

Performance of our approach on the channel layout planning

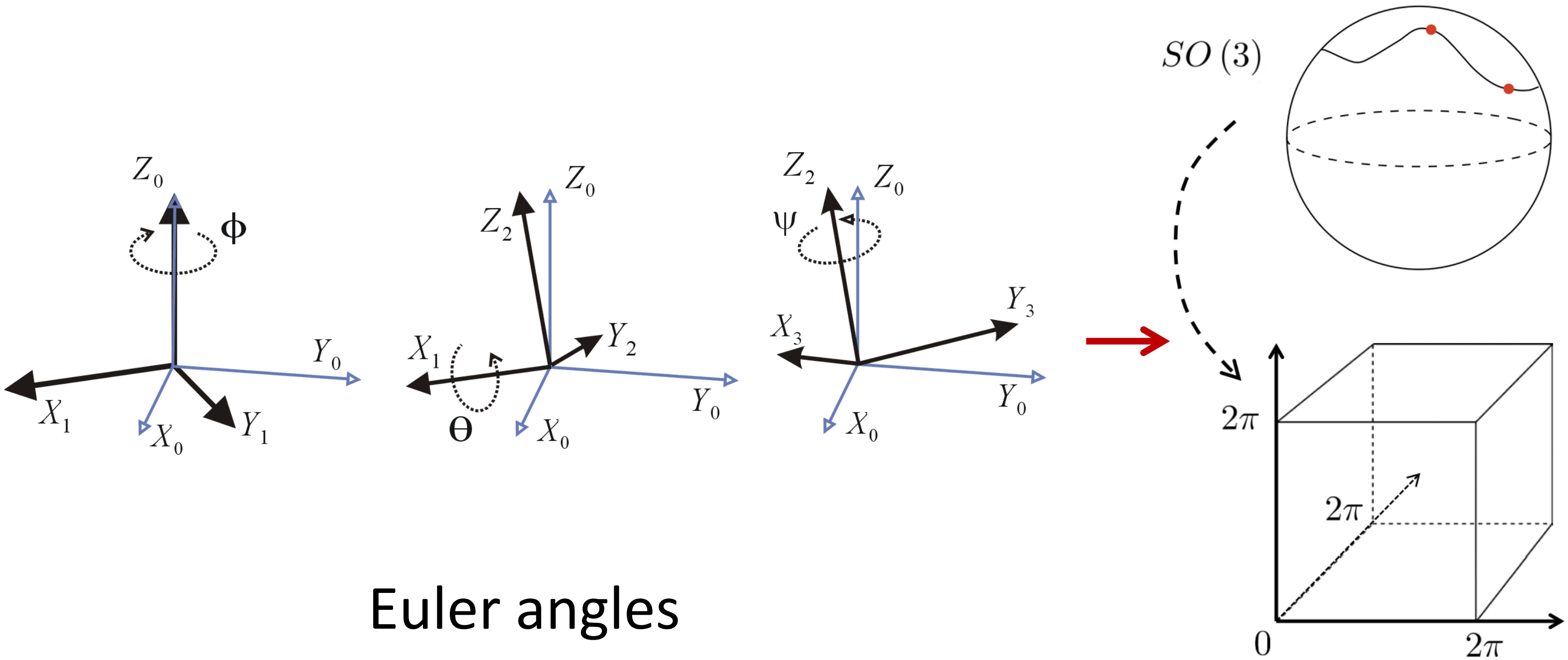


Takeaways

- Optimization over manifolds – Generalization of optimization over Euclidean spaces
- Define incremental parameterization and projection operators between tangent space and manifold
- Optimize over increments; reset after each SQP iteration!



Parameterization: Euler Angles



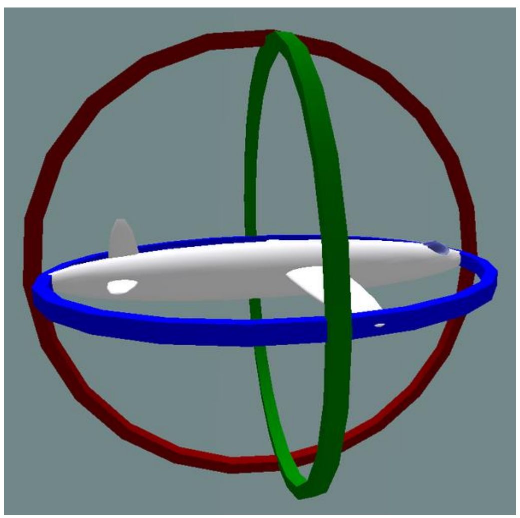
- Euler-Angle representation (cube)

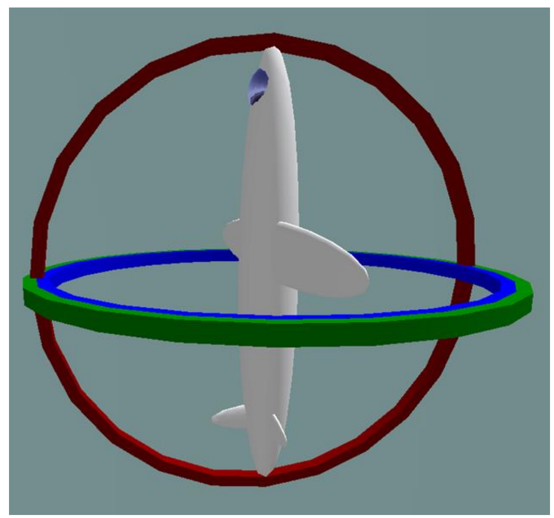
Euler angles

What problems do you foresee in directly using Euler angles in optimization?

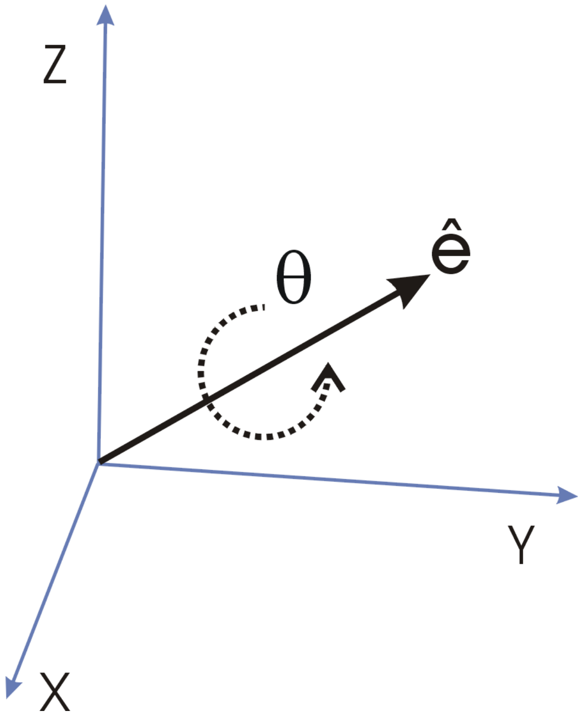
Parameterization: Euler Angles

- Topology not preserved: $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$
- Not unique, discontinuous
- Gimbal lock





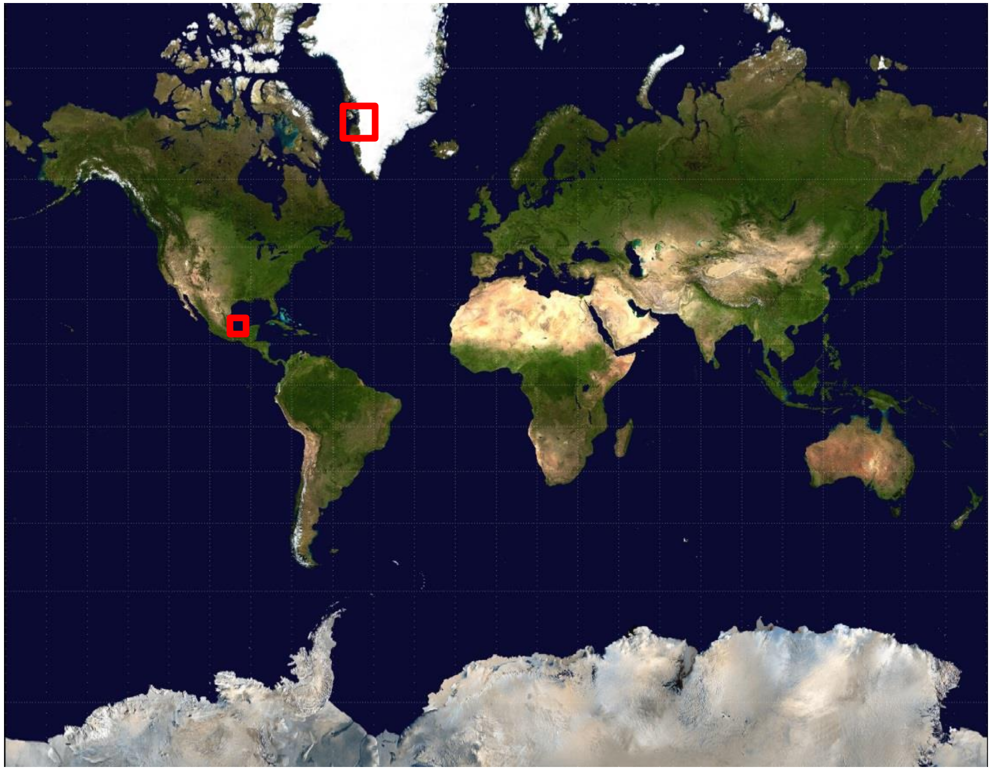
Parameterization: Axis-Angles



Orientation defined as rotation around axis

- 3-vector; norm of vector is the angle

Parameterization: Axis-Angles



Distances are not preserved!

Solution: Keep re-centering the axis-angle around a reference rotation (identity)