Today

Lagrange Multipliers.

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Fast Solution of Laplacian Systems.

Lagrangian Dual.

Find x, subject to

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$f_i(x) \leq 0, i = 1, \ldots m.$

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Remember calculus (constrained optimization.)

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$\lambda_i \geq 0$ - Lagrangian multiplier for inequality i.

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(B) positive for any λ.

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(A) there is no feasible x.

(B) there is no x, λ with $L(x, \lambda) < 0$.

Lagrangian:constrained optimization.

$$\begin{aligned} &\min \quad f(x) \\ &\text{subject to } f_i(x) \le 0, \qquad i = 1, ..., m \end{aligned}$$

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Lagrangian function:

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For all $\lambda \ge 0$ have $L(x,\lambda) \le v$

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Maximizing λ, only positive λ_i when $f_i(x) = 0$

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x, that minimizes $L(x,\lambda)$ over all $\lambda \geq 0$.

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Primal problem:
x, that minimizes $L(x, \lambda)$ over all $\lambda \geq 0$.

Dual problem:
λ, that maximizes $L(x, \lambda)$ over all x.

Why important: KKT.

Karash, Kuhn and Tucker Conditions.

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Launched nonlinear programming!

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$\min cx, Ax \geq b$

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Note: Lagrange multipliers for equality constraints.

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Usually: ν, and un-restricted.

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or

$L(\lambda, x) = -(\sum_j x_j (a_j \lambda - c_j)) + b\lambda$.

Best λ?

$\max b \cdot \lambda$ where $a_j \lambda = c_j$.

$\max b\lambda, \lambda^T A = c, \lambda \geq 0$

Duals!

Note: Lagrange multipliers for equality constraints.

Usually: ν, and un-restricted.

In this case, x for lagrangian of Dual.

Linear Systems...

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$\rightarrow$ symmetric diagonally dominant matrices by reduction.

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A graph $G = (V, E)$.

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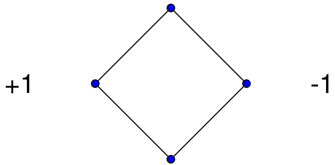

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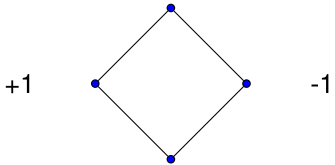

Claim: Minimizer is electrical flow.

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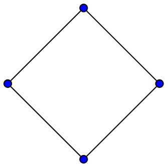
+1  -1

Claim: Minimizer is electrical flow.

Flow corresponds to flow induced by a set of potentials.

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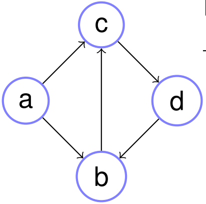

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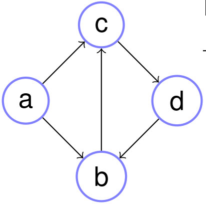

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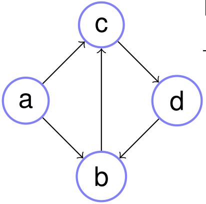

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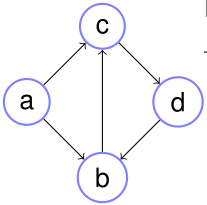

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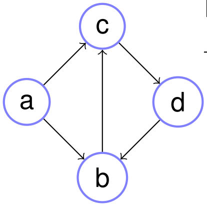

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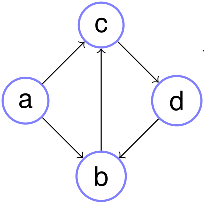

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Minimize Squared Potential differences!

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Algorithm: Work on flow and potentials.
To drive gap to 0.

Alg.

Given: χ, G

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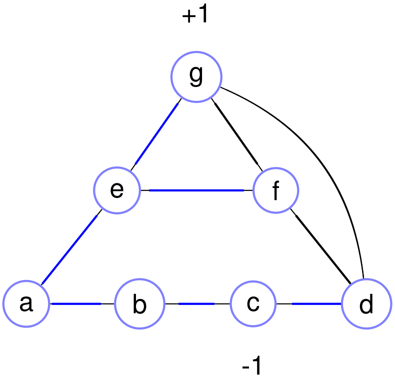

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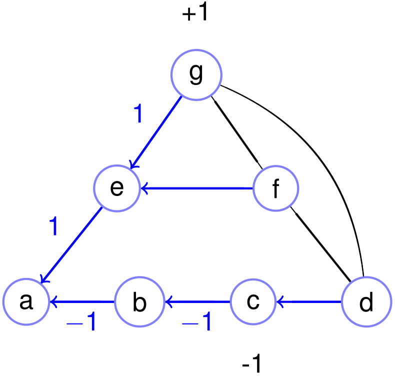
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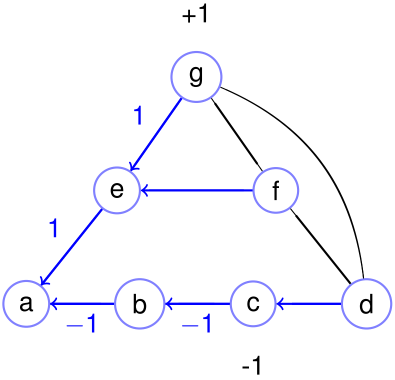

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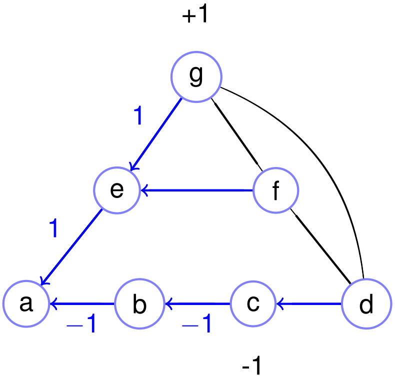
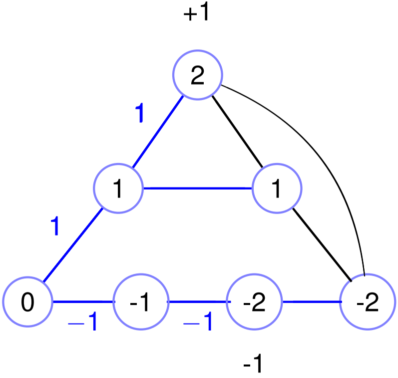
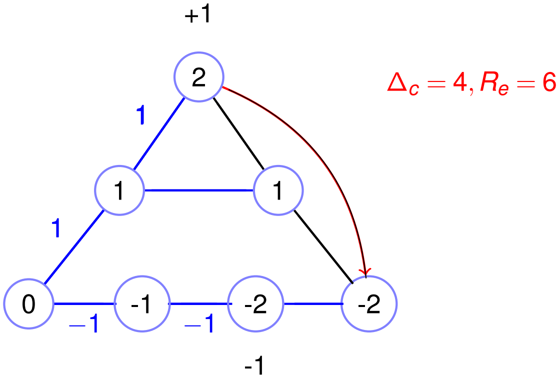
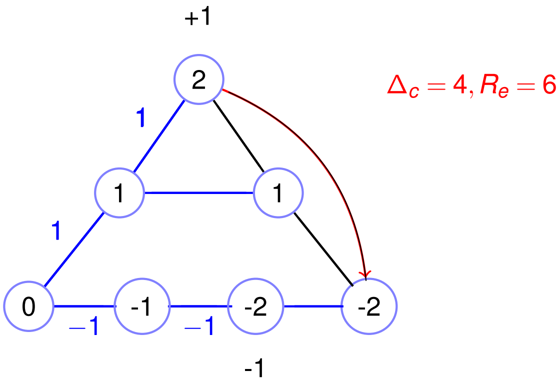

+1
g
1
e
f
1
a
b
c
d
−1
−1
-1

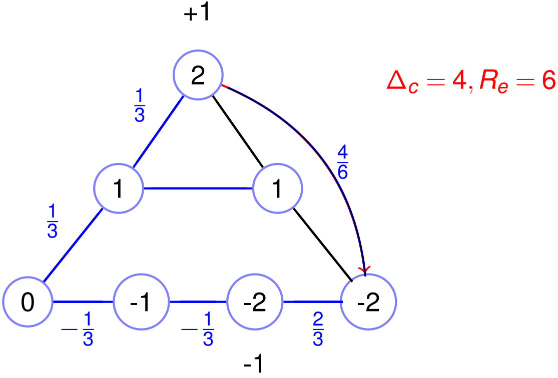


+1
2
1
1
1
1
0
-1
-2
-2
−1
−1
-1

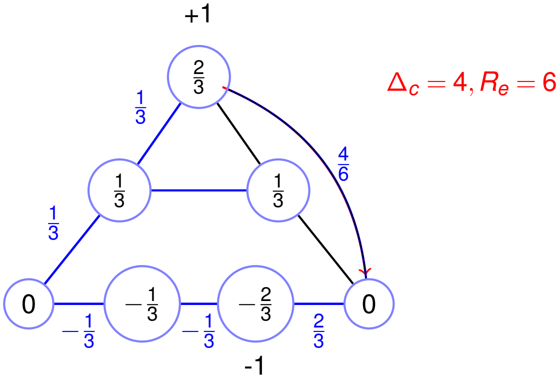


+1
2
1
1
1
1
$\Delta_c = 4, R_e = 6$
0
-1
-2
-2
-1
-1
-1


+1
2
1
1
1
1
0
-1
-2
-2
−1
−1
-1
$\Delta_c = 4, R_e = 6$



+1
2
1
1
0
-1
-2
-2
-1
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{4}{6}$
$-\frac{1}{3}$
$-\frac{1}{3}$
$\frac{2}{3}$
$\Delta_c = 4, R_e = 6$



+1
$\frac{2}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{4}{6}$
0
$-\frac{1}{3}$
$-\frac{2}{3}$
0
$-\frac{1}{3}$
$-\frac{1}{3}$
$\frac{2}{3}$
-1
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Energy reduction.

Given T,$e = (u,v)$, let $R_e = l_T(u,v) + 1$.

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(assume e' are oriented around cycle.)

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Difference in energy from f and f'.

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Fix $1/R_e$ of a cycle violation!

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Algorithm maintains feasible ϕ, f, $(B^T f = \chi)$

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Total distance from optimal is cycle violations!

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Claim: $E[\text{change in energy}|Gap] = \frac{Gap}{\tau}$ (τ is stretch of E in T.)

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Laplacian Systems are quite general: Climate, physics, SDD-matrices.

See you ...

Tuesday.