Today

Load balancing.

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Load balancing.

Balls in Bins.

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Balls in Bins.

Power of two choices.

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Load balancing.

Balls in Bins.

Power of two choices.

Cuckoo hashing.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$$

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$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdot 1}$

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \frac{n^k}{k!} \le \left(\frac{ne}{k}\right)^k$$

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdot 1} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1}$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{ne}{k}\right)^k$$

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$n(n-1)\cdots(n-k+1) \leq n^k$

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$$n(n-1)\cdots(n-k+1) \le n^k$$

$$k! \ge \left(\frac{k}{e}\right)^k$$

Simplest..

Load balance: m balls in n bins.

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For simplicity: n balls in n bins.

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Round robin:

Simplest..

Load balance: m balls in n bins.

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Round robin: load 1

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Centralized!

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Uniformly at random?

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Uniformly at random? Average load

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Max load?

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Max load?

n.

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Max load?

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Max load?

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Max load with probability $\geq 1 - \delta$?

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$\delta = \frac{1}{n^c}$ for today.

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Lemma: Max load is $\Theta(\log n)$ with probability $\geq 1 - \frac{1}{n}$.

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(W.h.p. - means with probability at least $1 - O(1/n^c)$ for today.)

Power of two..

n balls in n bins.

Power of two..

n balls in n bins.

Choose two bins, pick least loaded.

Power of two..

n balls in n bins.

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still distributed, but a bit less than not looking.

Power of two..

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Is max load lower?

Power of two..

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Is max load lower? Yes?

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How much lower?

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How much lower?

$\log n/2$?

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$\log n/2$? $\sqrt{\log n}$?

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$O(\log\log n)$!

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Exponentially better!

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Exponentially better! Old bound is exponential of new bound.

Analysis.

$n/8$ balls in n bins.

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Each ball chooses two bins at random.

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View as graph.

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View as graph.
Bin is vertex.

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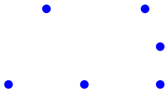
View as graph.
Bin is vertex.
Each ball is edge.



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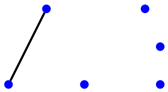

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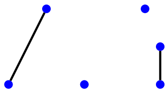

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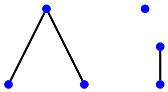
Analysis Intuition:



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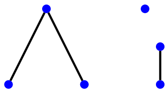

Analysis Intuition:
Add edge, add one to lower endpoint's "count."

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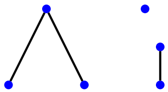

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Max load is max vertices count.

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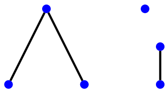

Analysis Intuition:
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Max load is max vertices count.
If max count is k.

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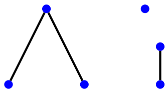

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Add edge, add one to lower endpoint's "count."

Max load is max vertices count.
If max count is k.
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$n/8$ balls in n bins.

Each ball chooses two bins at random.
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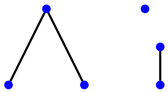

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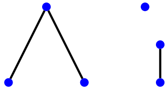

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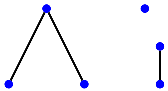

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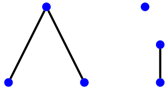

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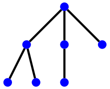

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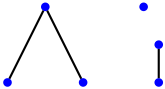

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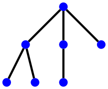

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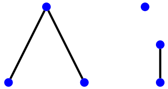

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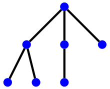

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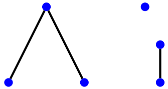

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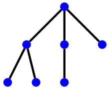

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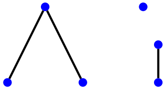

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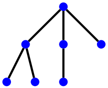

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Extend tree intuition.

Connected Component.

Claim: Component size in n vertex, $\frac{n}{8}$ edge random graph is $O(\log n)$ w/ prob. $\geq 1 - \frac{1}{n^c}$.
pause
Proof: Size k component, C, contains $\geq k-1$ edges.

$$\Pr[|C| \geq k] \leq \binom{n}{k}\binom{n/8}{k-1}\left(\frac{k}{n}\right)^{2(k-1)} \tag{1}$$

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Choose $k = -(c+1)\log_{.93} n$ make probability $\leq 1/n^c$.

Not dense.

Induced degree of node on subset, S, is degree of internal edges.

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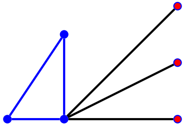

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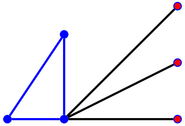

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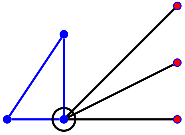

Induced degree of nodes in blue subset is 2, not 5!

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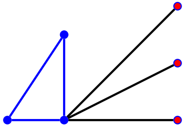

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Claim: Average induced degree on any subset of nodes is ≤ 8 with probability $\geq 1 - O(\frac{1}{n^2})$.

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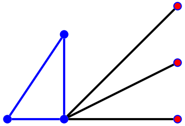

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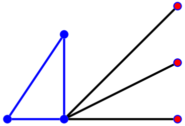

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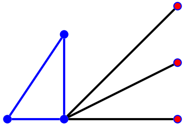

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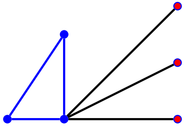

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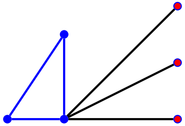

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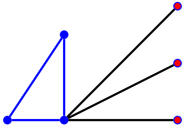

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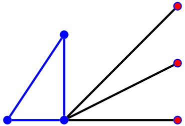

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Starts at $1/n^3$, decreasing till $k \leq n/8$ (at least)
$\rightarrow$ Total $O(1/n^2)$.

Removal Process!

Random Graph: Component size is $c \log n$ and max-induced degree is 8 w.h.p.

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Claim: $O(\log X)$ iterations where X is max component size.

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For any connected component:
Average induced degree 8 $\rightarrow$ half nodes w/degree ≤ 16.

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For any connected component:
Average induced degree 8 $\rightarrow$ half nodes w/degree ≤ 16.
$\rightarrow$ half nodes removed in each iteration.

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Recall edge corresponds to ball.

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Height of ball, h_i, is load of bin when it is placed in bin.

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$\rightarrow \log X$ iterations to remove all nodes.

Claim: Max load is $O(\log \log n)$ w.h.p.

Recall edge corresponds to ball.
Height of ball, h_i, is load of bin when it is placed in bin.
Corresponding edge removed in iteration r_i.

Removal Process!

Random Graph: Component size is $c \log n$ and max-induced degree is 8 w.h.p.

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Hashing with two choices: max load $O(\log \log n)$.

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$$\Pr[C_\ell] \leq \binom{m}{\ell}\binom{n}{\ell}\left(\frac{\ell}{n}\right)^{2(\ell)} \leq \left(\frac{e^2}{8}\right)^{\ell} \tag{3}$$

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Sum up

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See you on Thursday...