# Quantum Computing 2019 Set 1

Due September 12th

**Instructions: Solutions should be legibly handwritten or typset. Sets are to be returned in the mailbox outside 615 Soda Hall.**

**Problem 1** (Expectation of an operator)**.** In practice, we care about the outcome of a quantum system averaged over many trials. Consider a qubit $|\psi\rangle \in \mathbb{C}^2$ and associate the measurement $|0\rangle$ with $+1$ and a measurement of $|1\rangle$ with $-1$.

1. **(2 points)** Show the expectation of this experiment is $\langle\psi|Z|\psi\rangle$ where
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1| \,.$$

2. **(2 points)** This gives rise to the notation, $\langle Z\rangle_\psi = \langle\psi|Z|\psi\rangle$ (or $\langle Z\rangle$ when the state $\psi$ is clear from context). Give an experiment with expectation $\langle X\rangle$ where
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Problem 2** (CHSH Game)**. (4 points)** Recall the CHSH game discussed in the first lecture. In this game, Alice and Bob receive random inputs $x, y \in \{0,1\}$ and they win when they output $a, b \in \{0,1\}$ such that $a \oplus b = xy$. In other words they play to maximize the $Pr[a \oplus b = xy]$. Physicists often describe the same game in different notation. As before Alice and Bob receive random inputs $x, y \in \{0,1\}$, but now they output $u, v \in \{1, -1\}$. Denote by $u_x$ Alice's output when she receives $x$, and $v_y$ Bob?s output when he receives $y$. Then in this notation, the goal of the players is to maximize $S = E(u_0v_0) + E(u_0v_1) + E(u_1v_0) - E(u_1v_1)$. Make sure that you understand why the physics formulation is equivalent to the version discussed in class. In particular, show that the maximum value of $S$ for classical players is 2, whereas for quantum players it is at least
$$4(\cos^2 \pi/8 - \sin^2 \pi/8) = 2\sqrt{2}.$$

**Problem 3** (The GHZ game)**.** In this problem, we explore another game, like the CHSH game, that demonstrates this "spooky action" at a distance.

The setup of the game is the similar to that of the CHSH game, except there are 3 players Alice, Bob and Charlie. The referee will send each of the players as input the string "**x**" or the string "**y**" and expects in return a bit $\{+1, -1\}$. However, the referee will only give out inputs consisting of zero or two **y**'s; the 4 possible inputs are $\{\mathbf{xxx}, \mathbf{xyy}, \mathbf{yxy}, \mathbf{yyx}\}$. The players win if in the case that the input is **xxx**, the product of their outputs is $+1$ and in the case that two of them received **y** as input, the product of their outputs is $-1$. Equivalently,

$$\begin{aligned}
\text{input: } \mathbf{xxx} &\longrightarrow \text{output product: } +1 \\
\text{input: } \mathbf{xyy} &\longrightarrow \text{output product: } -1 \\
\text{input: } \mathbf{yxy} &\longrightarrow \text{output product: } -1 \\
\text{input: } \mathbf{yyx} &\longrightarrow \text{output product: } -1
\end{aligned}$$

1. **(2 points)** Show that if the 3 players, Alice, Bob and Charlie each employ a deterministic strategy, they cannot win with probability 1. (Hint: proof by contradiction). Give a simple deterministic strategy that wins with probability 3/4.

2. **(2 points)** Argue that even if Alice, Bob and Charlie share randomness, they still cannot win with probability 1.

3. **(2 points)** Now, we will come up with a quantum strategy that wins with probability 1. Suppose that Alice, Bob, and Charlie share one part of the tripartite cat state

   $$|\gamma\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}.$$

   On input **x**, assume each player measures their part of the cat state in the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ basis (the eigenstates of the $X$ operator).

   Show that this measurement strategy allows the players to win with probability 1 on input **xxx**.

4. **(2 points)** On input **y**, assume each player measures their part of the cat state in the $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$ basis (the eigenstates of the $Y = iXZ$ operator).

   Show that this measurement strategy allows the players to win with probability 1 on input **xyy**.

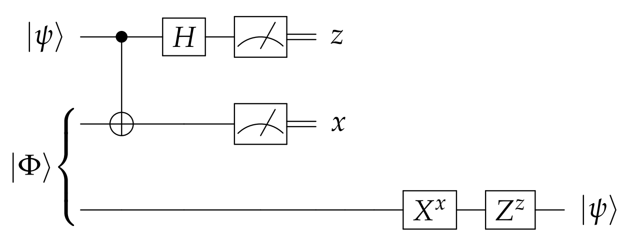
5. **(2 points)** Argue by symmetry that for the input cases **yxy** and **yyx**, the players win with probability 1.

**Problem 4** (Quantum teleportation)**.** Imagine Alice and Bob are on Earth and the Moon, respectively, and Alice wants to send to Bob her favorite qubit $|\psi\rangle$. Unfortunately, the only communication channel Alice and Bob have is classical; they can only send bits. However, before Bob left for the Moon, Alice and Bob came together to generate an EPR state,

$$|\Phi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

and each kept one half of the state. We will now construct a scheme in which Alice will perform measurements on $|\psi\rangle$ and her half of the EPR pair and send the measurements to Bob who can use the measurements to recover the state $|\psi\rangle$.

The following is the purported scheme as a quantum circuit. Alice controls the top two wires (the original state $|\psi\rangle$ and half of the EPR pair) and Bob controls the bottom wire. After Alice performs her measurements in the standard basis for outputs $x$ and $z$, she classically transports the bits $x$ and $z$ to Bob who applies gates conditionally. Here, we employ the convention that $X^1 = X$ and $X^0 = \mathbb{I}$.



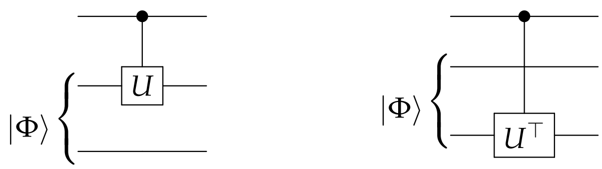

(1)

One way to analyze the correctness of this scheme is to write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and work through the unitaries and the measurements. Instead, we can use our understanding of delayed measurements and unitary multiplication to vastly simplify this analysis.

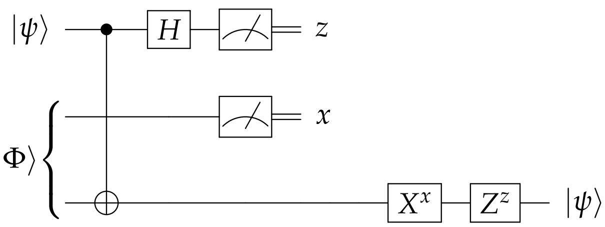
1. **(2 points)** Conclude that for an EPR state $|\Phi\rangle$ and any one qubit unitary $U$,

$$U \otimes \mathbb{I}\,|\Phi\rangle = \mathbb{I} \otimes U^\top\,|\Phi\rangle\,.$$

Show that this holds even if the unitary $U$ is controlled on a third qubit. I.e. show that the following two circuits are equivalent:



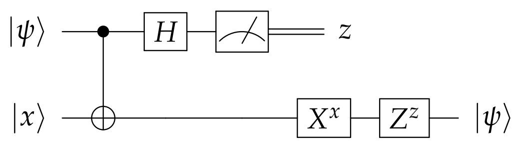

2. **(2 points)** Therefore, the following transformation of the circuit from (1) is valid:




(2)

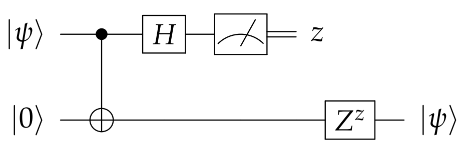
Notice that this circuit is no longer a teleportation circuit as Alice and Bob apply a shared quantum gate; this circuit is purely for analysis purposes.

Now using the principle of deferred measurement, argue that it is sufficient to consider the following circuit for all $x$.




(3)

3. **(2 points)** By commuting gates, argue that we can further simplify to only considering the following circuit.




(4)

4. **(2 points)** By expanding $|\psi\rangle$ as $\alpha\,|0\rangle + \beta\,|1\rangle$, show that (4) is valid.