An area law and sub-exponential algorithm for 1D systems

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Abstract

We give a new proof for the area law for general 1D gapped systems, which exponentially improves Hastings' famous result [1]. Specifically, we show that for a chain of d-dimensional spins, governed by a 1D local Hamiltonian with a spectral gap $\epsilon > 0$, the entanglement entropy of the ground state with respect to any cut in the chain is upper bounded by $\mathcal{O}(\frac{\log^3 d}{\epsilon})$. Our approach uses the framework of Refs. [2, 3] to construct a Chebyshev-based AGSP (Approximate Ground Space Projection) with favorable factors. However, our construction uses the Hamiltonian directly, instead of using the Detectability lemma, which allows us to work with general (frustrated) Hamiltonians, as well as slightly improving the $1/\epsilon$ dependence of the bound in Ref. [3]. To achieve that, we establish a new, "random-walk like", bound on the entanglement rank of an arbitrary power of a 1D Hamiltonian, which might be of independent interest: $\mathrm{ER}(H^\ell) \le (\ell d)^{\mathcal{O}(\sqrt{\ell})}$. Finally, treating d as a constant, our AGSP shows that the ground state is well approximated by a matrix product state with a sublinear bond dimension $B = e^{\tilde{\mathcal{O}}(\log^{3/4} n/\epsilon^{1/4})}$. Using this in conjunction with known dynamical programing algorithms, yields an algorithm for a $1/\mathrm{poly}(n)$ approximation of the ground energy with a subexponential running time $T \le \exp\left(e^{\tilde{\mathcal{O}}(\log^{3/4} n/\epsilon^{1/4})}\right)$.

1 Introduction

Understanding the structure and complexity of ground states of local Hamiltonians is one of the central problems in Condensed Matter Physics and Quantum Complexity Theory. In gapped systems, a remarkably general conjecture about the structure of ground states, *The Area Law*, bounds the entanglement that such states can exhibit. Specifically, for any subset S of particles, it bounds the entanglement entropy of ρ_S, the reduced density matrix of the ground state restricted to S, by the surface area of S, i.e., the number of local interactions between S and $\overline{S}$ [4].

Although the general area law remains an open conjecture, a lot of progress has been made on proving it for 1D systems. The breakthrough came with Hastings' result [1], which shows that the entanglement entropy across a cut for a 1D system is a constant independent of n, the number of particles in the system, and scales as $e^{\mathcal{O}(\frac{\log d}{\epsilon})}$, where d is the dimension of each particle and ϵ is the spectral gap. This result implies that the ground state of a gapped 1D Hamiltonian can be approximated in the complexity class NP.

In this paper, we:

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- Give an exponential improvement to $\tilde{\mathcal{O}}(\frac{\log^3 d}{\epsilon})$ in the bound of entanglement entropy for the general (frustrated) 1D Hamiltonians. The dependence on the gap even improves the previous best bound for frustration free 1D Hamiltonians and may possibly be tight to within log factors.

- Prove the existence of sublinear bond dimension Matrix Product State approximations of ground states for general 1D Hamiltonians. This implies a subexponential time algorithm for finding such states thus providing evidence that this task is not NP-hard.

We also establish the following properties of local Hamiltonians which may be of independent interest:

- "Random walk like" behavior of entanglement: for a 1D Hamiltonian H, the Entanglement Rank (ER) of H^ℓ is bounded by $(\ell d)^{O(\sqrt{\ell})}$.

- Let H' be the Hamiltonian consisting only of terms acting on a subset S of particles. Then the ground state of H has an exponentially small amount of norm in the "high" energy spectrum of H': the total norm with energy above t is $2^{-\Omega(t-|\partial S|)}$ where $|\partial S|$ is the size of the boundary between S and $\overline{S}$.

The work here has its origins in the combinatorial approach of [5], which used the Detectability lemma, introduced earlier in [6], to give a very different proof of Hastings' result for the special case of frustration-free Hamiltonians. The results there were greatly strengthened in [2] and [3], which introduced Chebyshev polynomials in conjunction with the detectability lemma to construct very strong AGSPs (approximate ground state projectors), leading to an exponential improvement of Hastings' bound in the frustration-free case to $O((\frac{\log d}{\epsilon})^3)$.

The starting point for our results is to consider a more general situation where the Hamiltonian obeys the 1D constraint only in a small neighborhood of s particles around the cut in question (see Fig. 1). The particles to the left and to the right of this small neighborhood are acted upon by multi-particle Hamiltonians H_L and H_R respectively. Constructing an AGSP for the new Hamiltonian is now much simpler, since the Hamiltonian has small norm: the AGSP is just a suitable Chebyshev polynomial of the Hamiltonian. In the frustration-free case, the new Hamiltonian has the same ground state as the original Hamiltonian, and this leads to a much simpler (and slightly stronger) proof of the Area Law. In the general, frustrated case, there is a tradeoff between the norm of the new Hamiltonian and how close its ground state is to that of the original Hamiltonian. To establish an area law, we must now consider a sequence of Hamiltonians whose ground states converge to the ground state of the original Hamiltonian, and derive an entropy bound from the tradeoff between the rate of convergence and the rate of increase of entanglement rank.

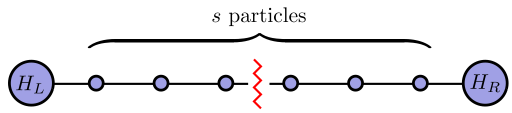

Figure 1: The 1D setting. We focus on a segment of s particles around the cut, denoting the multiparticle Hamiltonians to the left and right of the segment by H_L and H_R respectively.

2 Background: Approximate Ground State Projectors and their consequences

The overall strategy is to start with a product state $|\psi\rangle$ and repeatedly apply some operator K such that $\frac{1}{\|K^j|\psi\rangle\|}K^j|\psi\rangle$ approximates the ground state and the entanglement rank of $K^j|\psi\rangle$ is not too large. This property of an operator K is captured in the following definition of an approximate grounds state projection (AGSP):

Definition 2.1 (An Approximate Ground-Space Projection (AGSP)) *Consider a local Hamiltonian system $H = \sum_i H_i$ on a 1D chain, together with a cut between particles i^* and i^*+1 that bi-partitions the system. We say that an operator K is a (D,Δ)-Approximate Ground Space Projection (with respect to the cut) if the following holds:*

- ***Ground space invariance:*** *for any ground state $|\Gamma\rangle$, $K|\Gamma\rangle = |\Gamma\rangle$.*
- ***Shrinking:*** *for any state $|\Gamma^\perp\rangle \in \mathcal{H}^\perp$, also $K|\Gamma^\perp\rangle \in \mathcal{H}^\perp$, and $\||K|\Gamma^\perp\rangle\||^2 \leq \Delta$.*
- ***Entanglement:*** *the entanglement rank of K, as an element of the tensor product of two operator spaces (for the first and the second part of the system), is at most D.*

(The last condition implies that the operator K changes the entanglement rank of an arbitrary quantum state $|\phi\rangle$ at most by factor of D, i.e. $\mathrm{ER}(K|\phi\rangle) \leq D \cdot \mathrm{ER}(\phi)$*.)*

The parameters Δ and D capture the tradeoff between the rate of movement towards the ground state and the amount of entanglement that applying the operator K incurs. In [2, 3], it was shown that a favorable tradeoff gives an area law:

Lemma 2.2 *If there exists an (D,Δ)-AGSP with $D\cdot\Delta \leq \frac{1}{2}$, then there is a product state $|\phi\rangle$ whose overlap with the ground state is $\mu = |\langle\Gamma|\phi\rangle| \geq 1/\sqrt{2D}$.*

Lemma 2.3 *If there exists a product state whose overlap with the ground state is at least μ, together with a (D,Δ)-AGSP, then the entanglement entropy of $|\Gamma\rangle$ is bounded by*

$$S \leq \mathcal{O}(\frac{\log \mu^{-1}}{\log \Delta^{-1}}) \cdot \log D \ . \tag{1}$$

Combined, the above two lemmas give conditions for an area law:

Corollary 2.4 (Area Law) *If there exists an (D, Δ)-AGSP such that $D \cdot \Delta \leq \frac{1}{2}$, the ground state entropy is bounded by:*

$$S \leq \mathcal{O}(1) \cdot \log D \ . \tag{2}$$

3 Overview

The results here rely on the construction of a suitable AGSP that allows the application of Corollary 2.4. The first critical step is to exchange local structure far from the cut for a valuable **reduction in the norm** of the Hamiltonian. To do this, we isolate a neighborhood of $s+1$ particles around the cut in question, and then separately truncate the sum of the terms to the left and to the right of these $s+1$ particles. Specifically, we define the truncation of an operator as follows:

Definition 3.1 (Truncation) *For any self-adjoint operator A, form $A^{\leq t}$, the* truncation *of A, by keeping the eigenvectors the same, keeping the eigenvalues below $\leq t$ the same, and replacing any eigenvalue $\geq t$ with t.*

We then define $H^{(t)} = (\sum_{i<1} H_i)^{\leq t} + H_1 + \cdots + H_s + (\sum_{i>s} H_i)^{\leq t}$, where the s middle terms act on the the isolated string of $s+1$ particles around the cut. The result is a Hamiltonian H that is now norm bounded by $u = s + 2t$ acting on n particles with the following structure:

$$H = H^{(t)} = H_L + H_1 + H_2 + \cdots + H_s + H_R, \tag{3}$$

where each H_i are norm bounded by 1 and acts locally on particles $m+i$ and $m+i+1$, H_L acts on particles $1, \ldots, m$ and H_R acts on particles $m+s+1, \ldots, n$. We are interested in the entanglement entropy across the cut in the middle, i.e., between particles $m + s/2$ and $m + s/2 + 1$. In the frustration free case, it is clear that the ground state of $H^{(t)}$ is the same as that of the original Hamiltonian and it can be shown that the spectral gap is preserved for some constant value of t. For the frustrated case, the ground state of $H^{(t)}$ is no longer that of the original Hamiltonian and a limiting argument (see below) will be needed to complete the proof.

Having reduced the problem to a Hamiltonian with bounded norm u of the form (3), we turn to the next critical step of constructing the AGSP, the **use of Chebyshev polynomials** to approximate the projection onto the ground state. We begin with a suitably modified Chebyshev polynomial $C_\ell(x)$ of degree ℓ with the properties that $C_\ell(0) = 1$ and $|C_\ell(x)| \leq e^{-\Omega\left(\ell\sqrt{\epsilon/u}\right)}$ for $\epsilon \leq x \leq u$. The AGSP is then $K = C_\ell(H)$ and it is clear that $\Delta = e^{-\Omega\left(\ell\sqrt{\epsilon/u}\right)}$.

Bounding the ER for K requires important new ideas. We may take the approach of [2, 3] as a starting point and expand H^ℓ into terms of the form $H_{j_1} \cdots H_{j_\ell}$. For each such term, there is some i such that H_i occurs at most ℓ/s times. Thus, the entanglement rank of the given term across cut

i is less or equal to $d^{2\ell/s}$. The ER across the middle cut is at most times d^s times greater, which gives an upper bound $d^{2\ell/s+s}$. The difficulty is that the number of terms, $(s+2)^\ell$, is too large. To address this issue, we introduce formal commuting variables Z_i and consider the polynomial

$$P(Z) = (H_L Z_0 + H_1 Z_1 + \cdots + H_R Z_{s+1})^\ell = \sum_{a_0+\cdots+a_{s+1}=\ell} f_{a_0,\ldots,a_{s+1}} Z_0^{a_0} Z_1^{a_1} \ldots Z_{s+1}^{a_{s+1}}.$$

In particular, $H^\ell = \sum_{a_0,\ldots,a_{s+1}} f_{a_0,\ldots,a_{s+1}}$. This expression has fewer terms, namely, $\binom{\ell+s+1}{s+1}$. As before, for each multi-index $(a_0,\ldots,a_{s+1})$ there is some i such that $a_i \le l/s$. If i is fixed, a linear combination of the corresponding operators $f_{a_0,\ldots,a_{s+1}}$ can be generated as follows. We restrict our attention to only those terms in $P(Z)$ where Z_i appears at most ℓ/s times and assign arbitrary values to the variables $Z_0,\ldots,Z_{s+1}$. The ER of the resulting operator is estimated using the representation $P(Z) = (A + H_i Z_i + B)^\ell$, where A and B commute. We then use a **polynomial interpolation** argument to express each $f_{a_0,\ldots,a_{s+1}}$, their sum H^ℓ, and finally, the operator K. Thus we prove that the ER of K is at most $D = (dl)^{\mathcal{O}(l/s+s)}$.

Applying Theorem 2.4 to the above AGSP with $\ell = O(s^2)$, $s = \tilde{O}(\log^2(d)/\epsilon)$ yields our Area Law for frustration free Hamiltonians, providing an entanglement entropy bound of the form $\tilde{O}(\log^3(d)/\epsilon)$.

To address the frustrated case, a third critical result is needed: that the ground states of $H^{(t)}$ are very good approximations of the ground state of the original Hamiltonian. Intuitively, the structure of the small eigenvectors and eigenvalues of $H^{(t)}$ should approach those of H as t grows and we show that to be the case, showing a **robustness theorem**: that the ground states of $H^{(t)}$ and H are exponentially close in t and the spectral gaps are of the same order.

We would like to apply Theorem 2.4 to an AGSP for $H^{(t)}$, for t sufficiently large, however, if we try to do this in one step, the ER cost becomes a large function of t. Instead we use a well chosen arithmetic sequence $t_0, t_1, \ldots$ and the associated AGSP's to $H^{(t_i)}$ to guide the movement towards the ground state. The robustness theorem allows for very rapid convergence, the result of which is the area law in the general (frustrated) case.

4 Approximate Ground State Projector

Consider a Hamiltonian H acting on n particles with the following structure: $H = H_L + H_1 + H_2 + \cdots + H_s + H_R$, where H_i acts locally on particles $m+i$ and $m+i+1$, H_L acts on particles $1,\ldots,m$ and H_R acts on particles $m+s+1,\ldots,n$. Assume that H has a unique ground state $|\Gamma\rangle$ with energy ϵ_0 and that the other eigenvalues belong to the interval $[\epsilon_1, u]$. Let $\epsilon = \epsilon_1 - \epsilon_0$ denote the spectral gap. We wish to bound the entanglement entropy of $|\Gamma\rangle$ across the middle cut, $i = s/2$. (In our notation, cut i separates the particles $m+i$ and $m+i+1$.)

We define the AGSP as $K = C_\ell(H)$, where C_ℓ is a polynomial that satisfies the conditions below for a suitable value of Δ.

1. $C_\ell(\epsilon_0) = 1$;

2. $|C_\ell(x)| \le \sqrt{\Delta}$ for $\epsilon_1 \le x \le u$.

It follows that $K|\Gamma\rangle = |\Gamma\rangle$ and that the restriction of K to the orthogonal complement of $|\Gamma\rangle$ has norm less or equal to $\sqrt{\Delta}$.

Lemma 4.1 *There exists a degree ℓ polynomial C_ℓ that satisfies the above conditions for*

$$\sqrt{\Delta} = 2\, e^{-2\ell\sqrt{(\epsilon_1-\epsilon_0)/(u-\epsilon_0)}}.$$

Proof: We construct C_ℓ by a linear rescaling of the Chebyshev polynomial T_ℓ, which is defined by the equation $T_\ell(\cos\theta) = \cos(\ell\theta)$. It follows immediately that $|T_\ell(x)| \le 1$ for $x \in [-1,1]$. If $x > 1$, the equation $\cos\theta = x$ has a complex solution, $\theta = it$, where $\cosh t = x$. In this case, $T_\ell(x) = \cosh(\ell t) \ge \frac{1}{2}e^{\ell t}$. Since $t \ge 2\tanh(t/2) = 2\sqrt{(x-1)/(x+1)}$, we conclude that

$$T_\ell(x) \ge \frac{1}{2} e^{2\ell\sqrt{(x-1)/(x+1)}}.$$

Now, let

$$C_\ell(y) = \frac{T_\ell(f(y))}{T_\ell(f(\epsilon_0))}, \qquad \text{where} \quad f(y) = \frac{u+\epsilon_1-2y}{u-\epsilon_1}.$$

The function f maps ϵ_1 to 1 and u to -1, hence $|C_\ell(y)| \le \frac{1}{T_\ell(f(\epsilon_0))}$ for $y \in [\epsilon_1, u]$. The bound for $T_\ell(x)$ with $x = f(\epsilon_0)$ matches the expression for Δ because $\frac{\epsilon_1-\epsilon_0}{u-\epsilon_0} = \frac{f(\epsilon_1)-f(\epsilon_0)}{f(u)-f(\epsilon_0)} = \frac{x-1}{x+1}$. ■

Lemma 4.2 *The entanglement rank of $K = C_\ell(H)$ (where C_ℓ is an arbitrary degree ℓ polynomial) is bounded by $D = (d\ell)^{\mathcal{O}(\max\{\ell/s,\sqrt{\ell}\})}$.*

Proof: W.l.o.g. we may assume that $s \le \sqrt{\ell}$. If that is not the case, we can reduce s to $\sqrt{\ell}$ by joining some of the H_j's with either H_L or H_R. This does not change the actual entanglement rank or the required bound. After this reduction, the bound can be written as $(d\ell)^{\mathcal{O}(\ell/s)}$.

$K = C_\ell(H)$ is a linear combination of $\ell+1$ powers of H, and we will bound the entanglement rank added by each, focusing on the worst case H^ℓ. Let us first consider the expansion $H^\ell = \sum_{j_1,\ldots,j_\ell} H_{j_1}\ldots H_{j_\ell}$. It has too many terms to be useful, but we can group them by the number of occurrences of each H_j. To this end, we introduce a generating function, which is a polynomial in formal commuting variables $Z_0,\ldots,Z_{s+1}$:

$$P_\ell(Z) = (H_L Z_0 + H_1 Z_1 + \cdots + H_R Z_{s+1})^\ell = \sum_{a_0+\cdots+a_{s+1}=\ell} f_{a_0,\ldots,a_{s+1}} Z_0^{a_0} Z_1^{a_1} \ldots Z_{s+1}^{a_{s+1}}.$$

Each coefficient $f_{a_0,\ldots,a_{s+1}}$ is the sum of products $H_{j_1}\ldots H_{j_\ell}$, where each H_j occurs exactly a_j times. We are interested in estimating the ER of $H^\ell = \sum_{a_0,\ldots,a_{s+1}} f_{a_0,\ldots,a_{s+1}}$.

We start by noticing that for each multi-index $(a_0,\ldots,a_{s+1})$, there is some $i \in \{1,\ldots,s\}$ such that $a_i \le l/s$. Thus, $H^\ell = \sum_{i=1}^{s}\sum_{k=0}^{\ell/s} Q_{i,\ell k}$, where $Q_{i,\ell k}$ includes some of the operators $f_{a_0,\ldots,a_{s+1}}$

such that $a_i = k$ and $\sum_{j \neq i} a_j = \ell - k$. (This decomposition of H^ℓ is not unique.) We will, eventually, bound the ER of each $Q_{i,\ell k}$. To do that, we first define a generating function that includes all the matching $f_{a_0,\ldots,a_{s+1}}$'s:

$$P_{i,\ell k}(Z) = \sum_{\substack{a_i = k \\ \sum_{j\neq i} a_j = \ell - k}} f_{a_0,\ldots,a_{s+1}} \prod_{j \neq i} Z_j^{a_j}.$$

This sum has $t = \binom{\ell-k+s}{s}$ terms. The variable Z_i is excluded, or we may consider it equal to 1. When the remaining variables are assigned definite values, $Z \in \mathbb{C}^{s+1}$, we obtain a linear combination of the operators $f_{a_0,\ldots,a_{s+1}}$. The key observation is that such linear combinations have full rank, i.e. there are t distinct values of $Z \in \mathbb{C}^{s+1}$ such that the corresponding $\{P_{i,\ell k}(Z)\}$ form a basis in the space of operators of the form $\sum c_{a_0,\ldots,a_{s+1}} f_{a_0,\ldots,a_{s+1}}$, where $c_{a_0,\ldots,a_{s+1}} \in \mathbb{C}$ and the sum runs over the support of $P_{i,\ell k}$. In particular, $Q_{i,\ell k}$ is a linear combination of t operators of the form $P_{i,\ell k}(Z)$.

For a fixed Z, the operator $P_{i,\ell k}(Z)$ can be obtained as follows. We write $P_\ell(Z) = (A + H_i + B)^\ell$, where $A = \sum_{j<i} H_j Z_j$ and $B = \sum_{j>i} H_j Z_j$, and then collect the terms with H_i appearing exactly k times. Since A and B commute, such terms have the form $A^{a_0} B^{b_0} H_i \cdots H_i A^{a_k} B^{b_k}$. There are $\binom{\ell+k}{2k+1}$ distinct terms like that, and the ER of each term across cut i is at most d^{2k}. The ER across the middle cut is bounded by that number times $(d^2)^{|i-s/2|} \leq d^s$. Combining all factors, we find that

$$\mathrm{ER}(Q_{i,\ell k}) \leq \binom{\ell - k + s}{s} \binom{\ell + k}{2k+1} d^{2k+s} \leq \ell^{\mathcal{O}(s)} \ell^{\mathcal{O}(\ell/s)} d^{2\ell/s+s} \leq (d\ell)^{\mathcal{O}(\ell/s)}.$$

Here we have used the fact that $k \leq \ell/s$ and $s \leq \sqrt{\ell}$. The summation over i and k does not change this asymptotic form. ■

Lemma 4.2 gives a non-trivial tradeoff between the entanglement rank D and shrinking coefficient Δ of the operator K. By suitable choice of parameters this will give the desired (D, Δ)-AGSP such that $D \cdot \Delta \leq \frac{1}{2}$ and Corollary 2.4 will apply. One issue that we will have to address is the bound t on the norms of H_L and H_R. We first tackle the case of frustration free Hamiltonians, where we can assume W.L.O.G. that $t = O(1) = |H_L| = |H_R|$:

Let $H' = \sum H_i$ be a frustration free Hamiltonian with spectral gap ϵ. For t chosen in a moment, define $H_L = (\sum_{i \leq m} H_i)^{\leq t}$ and $H_R = (\sum_{i \geq m+s+1} H_i)^{\leq t}$ to be the truncation of the Hamiltonian acting on the left and right ends of the line. Set $H = H_L + H_1 + H_2 + \cdots + H_s + H_R$ so it is in the form as above. Clearly H has the same ground state as H'. Since $\epsilon_0 = 0$, Lemma 6.1 (below) yields that for $t = O(\frac{1}{\epsilon})$ the Hamiltonian H has a gap that is at least a constant times ϵ.

Theorem 4.3 *For a frustration free Hamiltonian $H' = \sum H_i$ with gap ϵ the entanglement entropy is $O(\frac{\log^3 d}{\epsilon})$.*

Proof: From Lemma 6.1, for $t = O(\frac{1}{\epsilon})$, we have that H has the same ground state and gap of the same order as H'. Recall lemma 4.2 applied to H describe an AGSP with bounds $\Delta = e^{-\frac{\ell\sqrt{\epsilon}}{\sqrt{s+2t}}}$ and

$D = (\ell+1)\binom{\ell+s}{s}^2(\frac{\ell}{s}+1)\binom{\ell+\frac{\ell}{s}}{2\frac{\ell}{s}}d^{2\ell/s}d^s$. Set $\ell = s^2/2$, $c = \sqrt{s}/\sqrt{s+2t}$ so that $\Delta = e^{-c\epsilon^{1/2}s^{3/2}}$, and $D \le \binom{(s^2+s)/2}{s}^4 d^{2s}$.

Write the condition $D\Delta < 1/2$ as $\log D < \log\frac{1}{\Delta} - 1$. $\log\frac{1}{\Delta} = c\epsilon^{1/2}s^{3/2}$, and $\log D = O(s(logd + \log(s^2+s)))$. Thus we can satisfy the condition with $s = O(\frac{log^2 d}{\epsilon})$, and therefore $\log D = O((\frac{log^3 d}{\epsilon}))$ and the result follows directly from Corollary 2.4.

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5 Low bond dimension MPS for frustration free 1D Hamiltonians

We can use these results to show the existence of a matrix product state of sub-linear bond dimension of size $\exp(O(\epsilon^{-\frac{1}{3}}\log^{\frac{2}{3}} n))$, that approximates a ground state $|\Gamma\rangle$ of a gapped frustration free 1D Hamiltonian to within $\frac{1}{poly(n)}$. To show the existence of a matrix product state of bond dimension B within δ of the $|\Gamma\rangle$, it suffices to show the existence of a state of entanglement rank B within $\frac{\delta}{n}$ of $|\Gamma\rangle$.

We've shown the existence of a state $|\psi\rangle$ with constant overlap with $|\Gamma\rangle$ and entanglement rank $O(\frac{\log^3 d}{\epsilon})$. To this state, we would like to apply an AGSP with $\Delta = \frac{1}{poly(n)}$. Just as in the proof of Theorem 4.3, we choose $\ell = s^2$ and Lemma 4.2 establishes that a $\Delta = e^{-c\epsilon^{-1/2}s^{3/2}}$, $D = exp(s\log d)$ AGSP exists. Setting $s = O(\epsilon^{-\frac{1}{3}}\log^{\frac{2}{3}} n)$, we have $\Delta = \frac{1}{poly(n)}$ with $D = \exp(O(\epsilon^{-\frac{1}{3}}\log^{\frac{2}{3}} n))$.

6 Frustrated Case

6.1 The operator $H^{(t)}$

We consider the ground state $|\Gamma\rangle$ of a local Hamiltonian $H' = \sum H'_i$, where H'_i acts locally on the particles i and $i+1$, and $0 \le H'_i \le \mathbb{1}$. We assume the Hamiltonian H' has a unique ground state with energy ϵ_0 and next lowest energy ϵ_1; let $\epsilon = \epsilon_1 - \epsilon_0$ denote the spectral gap. It is easy to see that in such a case $\epsilon \le 1$.

Ideally we wish to replace H' with some Hamiltonian H with the same ground state $|\Gamma\rangle$ and spectral gap ϵ, but with smaller norm, so that the AGSP from Section 4 yields a good bound on the entanglement entropy of $|\Gamma\rangle$. Towards that goal we consider Hamiltonians of the following more general form:

$$H = H_L + H_1 + H_2 + \cdots + H_s + H_R, \tag{4}$$

where H_i acts locally on particles $m+i$ and $m+i+1$, H_L acts on particles $1,\ldots,m$ and H_R acts on particles $m+s+1,\ldots,n$. We further require that H_L and H_R are positive and $|H_1 + \ldots H_{s-2}| \le s$.

We now show that $|\Gamma\rangle$ is the ground state of such a Hamiltonian H with the added properties that the ground energy of H_L, H_R and $\sum_{i=4}^{s-3} H_i$ are all 0 and $0 \leq H_i \leq 1$ for $i = 1, 2, 3, s-2, s-1, s$. By enforcing these properties, we have $\epsilon_0 \leq 6$. We do this by setting:

- $H_L = \sum_{i=1}^{m} H'_i - c\mathbb{1}$ where c is the ground energy of $\sum_{i=1}^{m} H'_i$,
- $H_R = \sum_{i=m+s+1}^{n} H'_i - c'\mathbb{1}$ where c' is the ground energy of $\sum_{i=m+s+1}^{n} H'_i$,
- $H_i = H'_{m+i}$ for the six values $i = 1, 2, 3, s-2, s-1, s$,
- $H_i = H_{m+i} - \frac{d}{s-7}\mathbb{1}$ for $4 \leq i \leq s-3$, where d is the ground energy of $\sum_{i=4}^{s-3} H'_i$.

It is easily verified that H is of the form (4), and since the difference between H and H' is a multiple of the identity, H has ground state $|\Gamma\rangle$. The Hamiltonian H has ground energy ≤ 6 since the tensor product of the ground states for the disjoint operators H_L, $\sum_{i=4}^{s-3} H_i$, and H_R only can have non-zero energy on $H_1 + H_2 + H_3 + H_{s-2} + H_{s-1} + H_s$.

To bound the norm (so as to effectively apply the AGSP from Section 4) we use the previously defined notion of truncation (Definition 3.1). If for any self-adjoint operator A, we let P_t be the projection into the subspace of eigenvectors of A with eigenvalues $\leq t$, then

$$A^{\leq t} = P_t A P_t + t(1 - P_t) \ . \tag{5}$$

Define $H^{(t)} = (H_L + H_1)^{\leq t} + H_2 + \cdots + H_{s-1} + (H_s + H^R)^{\leq t}$; it is easy to verify for $t \geq 0$, $H^{(t)} \leq H$ and $H^{(t)} \leq (2t + s)\mathbb{1}$. Unfortunately, the truncated Hamiltonian $H^{(t)}$ no longer has the same ground state $|\Gamma\rangle$. Intuitively, the structure of the small eigenvectors and eigenvalues of $H^{(t)}$ should approach those of H as t grows. The Robustness Theorem (Theorem 6.1), stated below and proved in Section 6.3, verifies this intuition, showing that for t bigger than some constant, the gap of $H^{(t)}$ is of the same order as the gap of H and the ground states of $H^{(t)}$ and H are exponentially close in t:

Theorem 6.1 (Robustness Theorem) *Let $|\Gamma\rangle, \epsilon_0, \epsilon_1$ be the ground state, the ground energy and the first excited level of H, and let $|\phi\rangle, \epsilon'_0, \epsilon'_1$ be the equivalent quantities of $H^{(t)}$. Then for $t \geq \mathcal{O}(\frac{1}{\epsilon_1 - \epsilon_0}(\frac{\epsilon_0}{\epsilon_1 - \epsilon_0} + 1))$, we have*

a. $\epsilon'_1 - \epsilon'_0 \geq \mathcal{O}(\epsilon_1 - \epsilon_0)$

b. $\| |\phi\rangle - |\Gamma\rangle \|^2 \leq 2^{-\mathcal{O}(t)}$.

6.2 General 1D Area Law

We would like to apply lemma 4.2 to $H^{(t)}$, for t sufficiently large, however, if we try to do this in one step, the entanglement rank cost becomes a large function of t. The overall plan is therefore to use a well chosen sequence $t_0, t_1, \ldots$ to guide the movement towards the ground state. More concretely we define a sequence of states $|\psi_0\rangle, |\psi_1\rangle, \dots$ that converge to $|\Gamma\rangle$, while carefully controlling the tradeoff between increase in entanglement rank and increase in overlap with $|\Gamma\rangle$.

Denote by $1-\mu_i$, the overlap between the ground state $|\phi^{t_i}\rangle$ of $H^{(t_i)}$ and $|\Gamma\rangle$. We use the AGSP $K = C_\ell(H^{(t_i)})$ from lemma 4.2 to move from state $|\psi_{i-1}\rangle$ to $|\psi_i\rangle$, where $|\psi_i\rangle$ has overlap at least $1-\mu_i$ with $|\phi^{t_i}\rangle$. We will show that the increase in entanglement rank of each move is small enough to bound the entanglement entropy of the limiting state which is the ground state of H.

We now put all the ingredients together to prove an area law for general 1D systems:

Theorem 6.2 *For any Hamiltonian of the form $H = H_L + H_1 + \cdots H_s + H_R$ with a spectral gap of ϵ, the entanglement entropy of the ground state across the $(s/2, s/2+1)$ cut is bounded by $O(\frac{\log^3 d}{\epsilon})$.*

We begin with a lemma:

Lemma 6.3 *There are constants t_0 and c and states $|\psi_i\rangle$ with entanglement rank R_i, $i = 0, 1, 2\dots$ satisfying:*

1. $|\langle\psi_i||\Gamma\rangle| \geq 1 - O(2^{-i}),\ i \geq 0.$
2. $\log R_0 = O(\frac{\log^3 d}{\epsilon})$, $\log R_i = \log R_0 + O(\sum_{j=1}^{i} \ell_j \log d)$, *with* $\ell_j = O(\sqrt{(t_j+s)/\epsilon})$.

Proof of Lemma:

We begin by choosing constants t_0, c such that $\Omega(t_0 + ic) \geq i + 4$, for the $\Omega(t)$ appearing in Theorem 6.1. Setting $t_i = t_0 + ic$, we therefore have

$$\left\| |\Gamma\rangle - |\phi^{t_i}\rangle \right\|^2 \leq 2^{-(i+4)} \tag{6}$$

for all i. Similar to the frustration free case, since t_0 is constant, choosing $\ell = s^2$ and $s = O(\frac{\log^2 d}{\epsilon})$ gives $D\Delta \leq \frac{1}{2}$ in Lemma 4.2 and thus by Lemma 2.2 there exists a product state $|\psi\rangle$ such that $|\langle\Gamma||\psi\rangle| \leq \frac{1}{\sqrt{2D}}$ where $\log D = O((\frac{log^3 d}{\epsilon}))$. Returning to Lemma 4.2, this time with ℓ chosen so that $\Delta = e^{-\frac{\ell\sqrt{\epsilon}}{\sqrt{s+2t_0-\epsilon_0}}} = O(\frac{1}{\sqrt{2D}})$, we establish that the state $|\psi_0\rangle = \frac{C_\ell(H_{\leq t_0})|\psi\rangle}{\|C_\ell(H_{\leq t_0})|\psi\rangle\|}$ has the property that $\left\| |\phi^{t_0}\rangle - |\psi_0\rangle \right\|^2 \leq \frac{1}{16}$, while having entanglement rank R_0 with $\log R_0 = O(\frac{\log^3 d}{\epsilon})$.

We now inductively define $|\psi_i\rangle$ from $|\psi_{i-1}\rangle$ and show that $\left\| |\phi^{t_i}\rangle - |\psi_i\rangle \right\|^2 \leq 2^{-i-4}$. Applying the triangle inequality to the induction hypothesis $\left\| |\phi^{t_{i-1}}\rangle - |\psi_{i-1}\rangle \right\|^2 \leq 2^{-i-3}$ along with the already established proximity of $|\phi^{t_{i-1}}\rangle, |\phi^{t_i}\rangle$ to $|\Gamma\rangle$ of (6), yields

$$\left\| |\phi^{t_i}\rangle - |\psi_{i-1}\rangle \right\|^2 \leq 2^{-i-1}. \tag{7}$$

Our goal is, with only a small amount of added entanglement, to move $|\psi_{i-1}\rangle$ a little bit closer to $|\phi^{t_i}\rangle$ which we will accomplish by using a well chosen AGSP. With $\ell_i = O(\sqrt{\frac{t_i+s}{\epsilon}})$, Lemma 4.2 establishes the existence of a $(D, \frac{1}{32})$ AGSP K for $H^{(t_i)}$ with a loose bound of $\log D \leq O(\ell_i \log d)$; we apply this AGSP K to move from $|\psi_{i-1}\rangle$ to $|\psi_i\rangle$ by setting $|\psi_i\rangle = \frac{K|\psi_{i-1}\rangle}{\|K|\psi_{i-1}\rangle\|}$. The shrinking property of the AGSP along with (7) establishes $\left\| |\phi^{t_i}\rangle - |\psi_i\rangle \right\|^2 \leq 2^{-i-4}$.

All told we have generated states $|\psi_i\rangle$ with entanglement rank $R_i = R_0 + \sum_{j\le i}\ell_i \log d$ and $\||\phi^{t_i}\rangle - |\psi_i\rangle\|^2 \le 2^{-i-4}$. Finally, $|\langle\psi_i|\Gamma\rangle| \ge 1 - \||\Gamma\rangle - |\psi_i\rangle\|^2/2 \ge 1-2^{-i}$ where the last inequality again used (6).

Proof of Theorem:

The above lemma gives a series of states of bounded entanglement rank that converge to the ground state $|\Gamma\rangle$. Thus if $\{\lambda_i\}$ are the Schmidt coefficients of Γ, Lemma 6.3

$$\sum_{i=1}^{R_i} \lambda_i^2 \ge |\langle\psi_i||\Gamma\rangle|^2 \ge 1 - O(2^{-i}).$$

The entropy of $|\Gamma\rangle$ is then upper bounded by summing $O(2^{-i})\log R_i$ (i.e. the maximal entropy contribution of mass $O(2^{-i})$ spread over $\log R_i$ terms). We arrive at a bound of the entanglement entropy of $|\Gamma\rangle$ given by:

$$\sum_i O(2^{-i})(\log R_0 + O(\sum_{j=1}^{i}\ell_j \log d)) = \log R_0 \sum_i O(2^{-i}) + \sum_i O(2^{-i}) \sum_{j=1}^{i} \sqrt{\frac{t_0+ic+s}{\epsilon}}\log d$$

$$= O(\log R_0) + O(\sqrt{\frac{t_0+s}{\epsilon}}) = O(\frac{\log^3(d)}{\epsilon}).$$

6.3 Proof of Theorem 6.1

Before giving the proof, we use a simple Markov bound to show that in a gapped situation, a state with low enough energy must be close to the ground state:

Lemma 6.4 (Markov) *Let B be a self-adjoint operator with lowest two eigenvalues $\epsilon_0 < \epsilon_1$; denote its lowest eigenvector by $|\psi\rangle$. Given a vector $|v\rangle$ with low energy, i.e. such that $\langle v|H|v\rangle \le \epsilon_0 + \delta$, then $|v\rangle$ is close to $|\psi\rangle$ in the following sense:*

$$\||\psi\rangle - |v\rangle\|^2 \le \frac{2\delta}{\epsilon_1 - \epsilon_0}\ .$$

Proof: Write $|v\rangle = a|\psi\rangle + \sqrt{1-a^2}|\psi^\perp\rangle$ where $|\psi^\perp\rangle$ is orthogonal to $|\psi\rangle$. The energy of $|v\rangle$ then satisfies

$$a^2\epsilon_0 + (1-a^2)\epsilon_1 \le \langle v|H|v\rangle \le \epsilon_0 + \delta,$$

and thus $(1-a^2) \le \frac{\delta}{\epsilon_1-\epsilon_0}$. The result follows from noting that $\||\psi\rangle - |v\rangle\|^2 = (1-a)^2 + (1-a^2) = 2-2a \le 2(1-a^2)$. ■

Proof of Theorem 6.1: Define A to be the sum of the two terms $A = H_2 + H_{s-1}$. Notice that the operators:

$$\{(H^{(t)} - A), H^L + H_1, H_s + H^R, H - A : t \ge 0\}$$

all commute with each other and therefore all the operators are simultaneously diagonalizable, i.e. they have a common collection of eigenstates. We fix P_t to be the projection onto the subspace spanned by those eigenstates of $H - A$ with eigenvalues less than t. This collection of eigenstates clearly have eigenvalues less than t for the operators $H^L + H_1$ and $H_s + H^R$ and therefore $H^L + H_1 = (H^L + H_1)^{\leq t}$ and $H_s + H^R = (H_s + H^R)^{\leq t}$ on the range of P_t. This allows the important observation that

$$H^{(t)} P_t = H P_t. \tag{8}$$

Proving part a:

The main idea is to consider the normalized projection of the two lowest eigenvectors of $H^{(t)}$ onto the lower part of the spectrum of $H - A$ using P_t. Under the assumption that t is sufficiently large and the gap of $H^{(t)}$ is sufficiently small (relative to the gap of H), we show contradictory facts about these normalized projections: that they are simultaneously far apart from each other (because applying P_t did not move either very much) and close to $|\Gamma\rangle$ (because they both have energy with respect to H that is close to ϵ_0). This contradiction allows us to conclude that for t sufficiently large, the gap of $H^{(t)}$ must be of the order of the gap of H.

For every normalized state $|v\rangle$, we define $|v_t\rangle = P_t|v\rangle$ and $|v_h\rangle = (1 - P_t)|v\rangle$. Note that by (8),

$$\langle v_t|H|v_t\rangle = \langle v_t|H^{(t)}|v_t\rangle \tag{9}$$

Our main technical tool is the following lemma that connects the energy of a state $|v\rangle$ to that of $|v_t\rangle$:

Lemma 6.5 *For any state $|v\rangle$,*

1. $\||v_h\rangle\| \leq \sqrt{\frac{\langle v|H^{(t)}|v\rangle}{t}}$,
2. $\langle v_t|H|v_t\rangle \leq \langle v|H|v\rangle + O(\sqrt{\frac{\langle v|H^{(t)}|v\rangle}{t}})$.

Proof: The first result follows from

$$\langle v|H^{(t)}|v\rangle \geq \langle v|(H^{(t)} - A)|v\rangle = \langle v_t|(H^{(t)} - A)|v_t\rangle + \langle v_h|(H^{(t)} - A)|v_h\rangle \geq \langle v_h|(H^{(t)} - A)|v_h\rangle \geq t\|v_h\|^2 ,$$

the last inequality by the definition of P_t.

For the second result,

$$\langle v_t|H|v_t\rangle = \langle v_t|H^{(t)}|v_t\rangle \leq \langle v|H^{(t)}|v\rangle + 2|\langle v_t|H^{(t)}|v_h\rangle|,$$

the first equality from (8) and the second inequality from writing $|v_t\rangle = |v\rangle - |v_h\rangle$ and expanding. We now bound the second term on the right hand side. Notice that $\langle v_t|H^{(t)}|v_h\rangle = \langle v_t|H^{(t)} - A|v_h\rangle + \langle v_t|A|v_h\rangle = \langle v_t|A|v_h\rangle$. To $|\langle v_t|A|v_h\rangle|^2$ we apply Cauchy-Schwartz to get $|\langle v_t|A|v_h\rangle|^2 \leq |A|v_t\rangle|^2||v_h\rangle|^2 \leq 2\sqrt{\frac{\langle v|H^{(t)}|v\rangle}{t}}$. ■

To prove part a, assume $\epsilon_1' - \epsilon_0' \leq \frac{1}{10}(\epsilon_1 - \epsilon_0)$ and denote by $|\phi^1\rangle$ the eigenvector of $H^{(t)}$ with eigenvalue ϵ_1'. Write $|\phi_t\rangle = P_t|\phi\rangle$, $|\phi_t^1\rangle = P_t|\phi^1\rangle$. Lemma 6.5 establishes

$$\langle\phi_t|H|\phi_t\rangle \leq \epsilon_0 + O(\sqrt{\frac{\epsilon_0}{t}})\ ,$$

$$\langle\phi_t^1|H|\phi_t^1\rangle \leq \epsilon_0 + \frac{1}{10}(\epsilon_1 - \epsilon_0) + O(\sqrt{\frac{\epsilon_0 + \frac{1}{10}(\epsilon_1 - \epsilon_0)}{t}}).$$

Setting $|v\rangle = \frac{|\phi_t\rangle}{\||\phi_t\rangle\|}$, $|v'\rangle = \frac{|\phi_t^1\rangle}{\||\phi_t^1\rangle\|}$, and using the above in Lemma 6.4 yields

$$\||\Gamma\rangle - |v\rangle\|^2 \leq O(1)\frac{\sqrt{\epsilon_0}}{\epsilon_1 - \epsilon_0}\frac{1}{\sqrt{t}},$$

$$\||\Gamma\rangle - |v'\rangle\|^2 \leq \frac{1}{10} + O(1)\frac{\sqrt{\epsilon_0 + \frac{1}{10}(\epsilon_1 - \epsilon_0)}}{\epsilon_1 - \epsilon_0}\frac{1}{\sqrt{t}}.$$

This establishes, for sufficiently large $t = \mathcal{O}(\frac{\epsilon_0 + \frac{1}{10}(\epsilon_1 - \epsilon_0)}{(\epsilon_1 - \epsilon_0)^2})$, that $|v\rangle$ and $|v'\rangle$ are both near $|\Gamma\rangle$ contradicting the fact that they are also almost orthogonal. ■

Proving part b:

We are interested in showing that the ground states of H and $H^{(t)}$ are very close together. Clearly, the ground states of the nearby Hamiltonians $H - A$ and $H^{(t)} - A$ are identical since they only differ among the eigenvectors with values above t and so the question becomes how much the addition of A can change things. This reduces to how much the operator A mixes the low and high spectral subspaces of $H - A$ (i.e. how big the off-diagonal contribution of A is when it is viewed in a basis that diagonalizes $H - A$). The core component of the argument will be the Truncation Lemma (Lemma 6.7): that the ground state $|\Gamma\rangle$ is exponentially close to the range of P_t (i.e. the low spectral subspace of $H - A$). We will combine this result with the fact that H and $H^{(t)}$ are identical on the range of P_t to argue that ground states for H and $H^{(t)}$ are exponentially close.

The following lemma captures the bounds necessary for proving the Truncation Lemma.

Lemma 6.6 *With H, P_t, $|\Gamma\rangle$, ϵ_0 as above we have the following:*

1. $\|(1 - P_t)|\Gamma\rangle\|^2 \leq \frac{2|\langle\Gamma|(1-P_t)AP_t|\Gamma\rangle|}{t-\epsilon_0}$,
2. *For* $t \geq u$, $\|(1 - P_t)HP_u\| = \|(1 - P_t)AP_u\| \leq 2e^{-\frac{t-u}{8}}$.

Proof: For 1., by definition,

$$\epsilon = \langle\Gamma|H|\Gamma\rangle = \langle\Gamma|P_tHP_t|\Gamma\rangle + \langle\Gamma|(1 - P_t)H(1 - P_t)|\Gamma\rangle + \langle\Gamma|P_tH(1 - P_t)|\Gamma\rangle + \langle\Gamma|(1 - P_t)HP_t|\Gamma\rangle.$$

This gives the bound

$$\epsilon \geq \epsilon \| P_t|\Gamma\rangle\|^2 + t\|(1-P_t)|\Gamma\rangle\|^2 - 2|\langle\Gamma|(1-P_t)AP_t|\Gamma\rangle|,$$

where the second term of the right hand side follows from the inequality $\langle\Gamma|(1-P_t)H(1-P_t)|\Gamma\rangle \leq \langle\Gamma|(1-P_t)(H-A)(1-P_t)|\Gamma\rangle$, and the replacement of H with A in the third term follows from $(1-P_t)(H-A)P_t = (1-P_t)P_t(H-A) = 0$. Writing $\|P_t|\Gamma\rangle\|^2 = 1 - \|(1-P_t)|\Gamma\rangle\|^2$ and rearranging terms yields statement 1.

For statement 2., the first inequality follows simply from writing $H = (H-A)+A$ and noting that $H-A$ commutes with P_u. We write $(1-P_t)AP_u = (1-P_t)e^{-r(H-A)}e^{r(H-A)}Ae^{-r(H-A)}e^{r(H-A)}P_u$, for an $r > 0$ to be chosen later, and noting therefore that

$$\|(1-P_t)AP_u\| \leq \|(1-P_t)e^{-r(H-A)}\| \cdot \|e^{r(H-A)}Ae^{-r(H-A)}\| \leq e^{-r(t-u)}\|e^{r(H-A)}Ae^{-r(H-A)}\| \ .$$

The Hadamard Lemma gives the expansion

$$\begin{aligned} e^{r(H-A)}Ae^{-r(H-A)} &= A + r[H-A,A] + \frac{r^2}{2!}[H-A,[H-A,A]] \\ &+ \frac{r^3}{3!}[H-A,[H-A,[H-A,A]]] + \cdots \\ &= Q_0 + rQ_1 + \frac{r^2}{2!}Q_2 + \frac{r^3}{3!}Q_3 + \cdots \ , \end{aligned} \tag{10}$$

and we turn to bounding the norm of these operators Q_i.

If we expand $H-A$ and A as the sum of its constituent local terms H_j, each Q_i can be written as a sum of n_i terms, each a product of H_j's; we now bound n_i. Notice that $n_0 = 2$ and that Q_{i-1} consists of terms, each of which is a product of at most i H_j's. For such a product, there are at most $2i$ terms in $H-A$ that do not commute with it. This implies the recursive bound $n_i \leq 4in_{i-1}$ and thus $n_i \leq 2 \cdot 4^i i!$. Since each of the terms is norm bounded by 1, we have $\|Q_i\| \leq 2 \cdot 4^i i!$. Plugging this bound into (10) we have $e^{r(H-A)}Ae^{-r(H-A)} \leq 2\sum_i (4r)^i$; choosing $r = \frac{1}{8}$ gives a bound of 4 for (10) and establishes statement 2. ■

Lemma 6.7 (Truncation Lemma) *For $t > 17$,*

$$\|(1-P_t)|\Gamma\rangle\| \leq 2^{-\Omega(t)}. \tag{11}$$

Proof: We show a discrete version of (11): that there exist constants $s = 16 + \epsilon_0$ and $d = 16$ such that for integers $n \geq 0$

$$\|(1-P_{s+nd})|\Gamma\rangle\| \leq 2^{-n} \tag{12}$$

The result will then follow since $s + nd \leq t \leq s + (n+1)d$ implies

$$\|(1-P_t)|\Gamma\rangle\| \leq \|(1-P_{s+nd})|\Gamma\rangle\| \leq 2^{-n} \leq 2^{-(\frac{t-s}{d}-1)} = 2^{-\Omega(t)}.$$

To prove (12), we will proceed by induction. Clearly the initial case of $n = 0$ holds. Assume that $\|(1 - P_{s+nd})|\Gamma\rangle\| \leq 2^{-n}$, for $n < n_0$. Define $P_{[0]} = P_s$ and $P_{[j]} = P_{s+jd} - P_{s+(j-1)d}$ for $1 \leq j \leq n_0$; thus $P_{s+n_0 d} = \sum_{j=0}^{n_0} P_{[j]}$ and the induction hypothesis implies

$$\|P_{[j]}|\Gamma\rangle\| \leq 2^{-j+1}. \tag{13}$$

for $j < n_0$. By Lemma 6.6,

$$\|(1 - P_{s+n_0 d})|\Gamma\rangle\|^2 \leq \frac{2|\langle\Gamma|(1 - P_{s+n_0 d})AP_{s+n_0 d}|\Gamma\rangle|}{s + n_0 d - \epsilon_0}.$$

Our goal is to bound the numerator of the right hand side by $16 \cdot 2^{-2n_0}$; (12) then follows since the denominator is at least 16. Write

$$|\langle\Gamma|(1 - P_{s+n_0 d})AP_{s+n_0 d}|\Gamma\rangle| = |\sum_{j=0}^{n_0} \langle\Gamma|(1 - P_{s+n_0 d})AP_{[j]}|\Gamma\rangle|$$

$$\leq \sum_{j=0}^{n_0} \|(1 - P_{s+n_0 d})AP_{[j]}\| \|(1 - P_{s+n_0 d})|\Gamma\rangle\| \|P_{[j]}|\Gamma\rangle\| \leq \sum_{j=0}^{n_0} \|(1 - P_{s+n_0 d})AP_{[j]}\| 2^{-j-n_0+2},$$

where the last equation used (13) and the fact that $\|(1 - P_{s+n_0 d})|\Gamma\rangle\| \leq \|(1 - P_{s+(n_0-1)d})|\Gamma\rangle\| \leq 2^{-(n_0-1)}$. Applying Lemma 6.6 to bound the first term in the sum on the right hand side yields:

$$|\langle\Gamma|(1 - P_{s+n_0 d})AP_{s+n_0 d}|\Gamma\rangle| \leq \sum_{j=0}^{n_0} e^{-\frac{(n_0-j)d}{8}} 2^{-(n_0+j-3)} = 2^{-2n_0}(8) \sum_{j=0}^{n_0} 2^{(n_0-j)(1-d\frac{1}{8\ln 2})}.$$

With the choice of $d = 16 \ln 2$, the sum in the last term on the right hand side is a geometric series that is bounded by 2 which yields the desired bound of $16 \cdot 2^{-2n_0}$ and completes the proof of (12). ■

We now use the Truncation Lemma to show that the the projected state $P_t|\Gamma\rangle$ is exponentially close to eigenvectors of both H and $H^{(t)}$.

Lemma 6.8 *The state* $|\Gamma_t\rangle = \frac{1}{\|P_t|\Gamma\rangle\|} P_t|\Gamma\rangle$ *is an approximate eigenvector of both* H *and* $H^{(t)}$ *in the following sense:*

$$\|H|\Gamma_t\rangle - \epsilon_0|\Gamma_t\rangle\| = \|H^{(t)}|\Gamma_t\rangle - \epsilon_0|\Gamma_t\rangle\| \leq 2^{-\Omega(t)}. \tag{14}$$

Proof: We begin by writing $H(1 - P_t)|\Gamma\rangle = \sum_{i=0}^{\infty} H(P_{t+i+1} - P_{t+i})|\Gamma\rangle$ and thus

$$\|H(1 - P_t)|\Gamma\rangle\| \leq \sum_{i=0}^{\infty} \|H(P_{t+i+1} - P_{t+i})\| \cdot \|P_{t+i}|\Gamma\rangle\| \leq \sum_{i=0}^{\infty} (t + (i+1) + 2) 2^{-\Omega(t+i)} \leq 2^{-\Omega(t)} ,$$

where the bound on the first term follows from the fact that $H - A \leq (t + i + 1) \cdot \mathbb{1}$ on the range of P_{t+i+1} and the bound on the second term from the Truncation Lemma.

We write $\epsilon_0|\Gamma\rangle = H|\Gamma\rangle = H(1-P_t)|\Gamma\rangle + HP_t|\Gamma\rangle$. We have bounded the first term on the right hand side by $2^{-\Omega(t)}$ and it follows simply that

$$\|HP_t|\Gamma\rangle - \epsilon_0|\Gamma\rangle\| = \|H^{(t)}P_t|\Gamma\rangle - \epsilon_0|\Gamma\rangle\| \leq 2^{-\Omega(t)}, \tag{15}$$

where the first equality is from (8). Multiplying (15) by $\frac{1}{\|P_t|\Gamma\rangle\|} \leq \frac{1}{\sqrt{1-2^{-\Omega(t)}}}$, a constant close to one yields (14) since the constant can be absorbed into $2^{-\Omega(t)}$. ■

We outline the remainder of the argument. The approximate eigenvalue property of (14) can be used to show that $|\Gamma_t\rangle$ is close to an eigenvector of $H^{(t)}$ with eigenvalue in the range $[\epsilon_0 - 2^{-\Omega(t)}, \epsilon_0 + 2^{-\Omega(t)}]$. By combining a lower bound for the ground energy of $H^{(t)}$ with the fact (part a.) that $H^{(t)}$ has a gap of reasonable size, we are able to show that the eigenvector of $H^{(t)}$ near $|\Gamma_t\rangle$ is the ground state of $H^{(t)}$. The proximity of $|\Gamma_t\rangle$ to the ground states of both $H^{(t)}$ and H then establishes the result.

We begin by setting $\delta \stackrel{\text{def}}{=} 2^{-\Omega(t)}$. The approximate eigenvalue property of (14) implies that there is an eigenvalue of $H^{(t)}$ within δ of ϵ_0, for if not, writing $|\Gamma_t\rangle = \sum_i c_i|v_i\rangle$ where $|v_i\rangle$ are the eigenvectors of $H^{(t)}$ with eigenvalues λ_i, $\|H^{(t)}|\Gamma_t\rangle - \epsilon_0\Omega_t\|^2 = \sum_i(\lambda_i - \epsilon_0)c_i^2 > \delta^2\sum_i c_i^2 = \delta^2$ which contradicts (14). We now show that there is only one eigenvalue in this range and that it is in fact the ground energy for $H^{(t)}$. This will follow by lower bounding the energy of $H^{(t)}$ by $\epsilon_0 + \delta - (\epsilon_1' - \epsilon_0')$.

Decompose an arbitrary state $|\psi\rangle = P_t|\psi\rangle + (1-P_t)|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$. Then

$$\langle\psi|H^{(t)}|\psi\rangle = \langle\psi|(H^{(t)} - A)|\psi\rangle + \langle\psi|A|\psi\rangle = \langle\psi_1|(H^{(t)} - A)|\psi_1\rangle + \langle\psi_2|(H^{(t)} - A)|\psi_2\rangle + \langle\psi|A|\psi\rangle \geq t\|\psi_2\|^2.$$

A $|\psi\rangle$ for which $t\|\psi_2\|^2 \geq \frac{\epsilon_0}{t}$ will therefore have energy at least ϵ_0. In the remaining case of $\||\psi_1\rangle\|^2 \geq 1 - \frac{\epsilon_0}{t}$ we bound the energy of $|\psi\rangle$ as follows:

$$\langle\psi|H^{(t)}|\psi\rangle \geq \langle\psi_1|H^{(t)}|\psi_1\rangle + \langle\psi_2|H^{(t)}|\psi_2\rangle - 2|\langle\psi_1|A|\psi_2\rangle| \geq \langle\psi_1|H^{(t)}|\psi_1\rangle + \langle\psi_2|H^{(t)}|\psi_2\rangle - 4\sqrt{\frac{\epsilon_0}{t}}. \tag{16}$$

Since $|\psi_1\rangle$ is in the range of P_t, (8) implies $\langle\psi_1|H^{(t)}|\psi_1\rangle = \langle\psi_1|H|\psi_1\rangle \geq \epsilon_0\|\psi\|^2$. Combining this bound with (16) gives $\langle\psi|H^{(t)}|\psi\rangle \geq \epsilon_0 - O(\frac{\epsilon_0}{t})$ and thus we've shown $\epsilon_0' \geq \epsilon_0 - O(\frac{\epsilon_0}{t})$. Since (from part a.) $\epsilon_1' - \epsilon_0' \geq O(\epsilon_1 - \epsilon_0)$, a choice of $t = O(\frac{\epsilon_0}{(\epsilon_1 - \epsilon_0)^2})$ ensures that the the ground energy of $H^{(t)}$ is at least $\epsilon_0 + \delta - (\epsilon_1' - \epsilon_0')$.

We've established that both ground energy and the energy of $|\Gamma_t\rangle$ with respect to $H^{(t)}$ is in the interval $[\epsilon_0 - \delta, \epsilon_0 + \delta]$. Applying Lemma 6.4 gives

$$\||\psi\rangle - |\Gamma_t\rangle\| \leq \delta,$$

and b. then follows from recalling that $\||\Gamma\rangle - |\Gamma_t\rangle\| \leq \delta$ as well.

7 Sub exponential algorithm for finding the ground energy of gapped 1D Hamiltonians

We now show that the ground state can be well-approximated by a MPS with a sublinear bond dimension, and, consequently, a $1/\text{poly}(n)$ approximation of its ground energy can be found in a subexponential time. To simplify the discussion, we shall treat d as a constant.

As discussed in the frustration free case, to show that $|\Gamma\rangle$ can be approximated to within $\frac{1}{poly(n)}$ with an MPS of bond dimension $B = \tilde{O}(\exp(\log^{\frac{3}{4}} n/\epsilon^{\frac{1}{4}}))$, it suffices to show for each cut $(i, i+1)$ the existence of a state with entanglement rank B across that cut that is within $\frac{1}{poly(n)}$ of $|\Gamma\rangle$. Theorem 6.1 yields that for $t = O(\log n)$, $\||\phi^{(t)}\rangle - \Gamma\|\| \leq \frac{1}{poly(n)}$, and therefore we turn to finding a state with entanglement rank B across the cut $(i, i+1)$ that approximates $|\phi^{(t)}\rangle$. Applying the AGSP $K = C_\ell(H^{(t)})$ of Lemma 4.2, we have $\Delta = \frac{1}{poly(n)}$ for $\ell = O(\log n\sqrt{\frac{s+\log n}{\epsilon}}))$, and $D = \tilde{O}(\exp(\ell/s + s))$. The optimal choice of $s = \log^{\frac{3}{4}} n/\epsilon^{\frac{1}{4}}$ gives the desired $B = e^{\tilde{O}(\log^{\frac{3}{4}} n/\epsilon^{1/4})}$.

We can now use this result to bound the complexity of actually finding the ground energy. For simplicity, we treat d, ϵ as constants. Using recent dynamical programing results [**?**, **?**], we infer that there exists an algorithm that runs in time $T = (dBn)^{\mathcal{O}(B^2)} \leq \exp(e^{\tilde{\mathcal{O}}(\log^{3/4} n/\epsilon^{1/4})})$ and finds a $1/\text{poly}(n)$ approximation of the ground energy. Since $e^{\tilde{\mathcal{O}}(\log^{3/4} n/\epsilon^{1/4})}$ is smaller than any finite root of n, it follows that

Corollary 7.1 *Finding a* $1/\text{poly}(n)$ *approximation to the ground energy of a 1D, nearest-neighbors Hamiltonian with a constant spectral gap is not* NP*-hard, unless 3-SAT can be solved in a sub-exponential time.*

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