Circumscribing an Ellipsoid about the Intersection of Two Ellipsoids

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This paper is an update, more readily accessible via the web and a little easier to read, I hope, of a paper with the same title the author first published in pp. 437-441 of the *Canadian Math. Bulletin* **11** #3 (1968) when he was on the faculty of the Mathematics and Computer Science Departments at the University of Toronto.

Abstract

The nontrivial intersection of two n-dimensional ellipsoids with a common center is so *Tightly* circumscribed by each ellipsoid in a specified one-parameter family of them that no other ellipsoidal surface can slip between the intersection and any ellipsoidal surface in that family.

Introduction

Among the bodies that may be chosen to circumscribe complicated regions by simple ones in a computer are ellipsoids. They are deemed "simple" because each can be represented by one inequality. However, often a complicated region turns out to be far smaller than any simple body circumscribed about it. Consequently computer programs may have to manipulate combinations like unions, intersections and sums of simple bodies. Storage capacity and the time consumed by computations limit the complexity achievable in practice, forcing occasional simplifications of which one kind is the replacement of the intersection of circumscribing bodies by one simpler circumscribing body, preferably not too much too big. This work's bodies are all ellipsoids, a few circumscribing the intersection of others as tightly as is possible and not too much too big.

Each n-dimensional solid ellipsoid $\mathbb{W}$ shall be identified with an n-by-n real symmetric positive (semi)definite matrix of the same name W via the relationship

$$x \in \mathbb{W} \quad \text{if and only if} \quad x'\cdot W\cdot x \le 1 .$$

Here row x' is the transpose of real column vector x . Boundary $\partial\mathbb{W}$ consists of all x for which $x'\cdot W\cdot x = 1$. Note that all ellipsoids discussed hereunder are centered at the origin o , and none of them can be *Flat* (contained in a proper subspace). But when W is not positive definite (not invertible) then $\mathbb{W}$ is a slab or an infinitely long cylinder with an ellipsoidal cross-section.

We seek a formula for a matrix H whose ellipsoid $\mathbb{H}$ contains the intersection $\cap_k \mathbb{M}_k$ of a finite collection of ellipsoids $\mathbb{M}_k$ given their respective matrices M_k . A formula comes to mind:

- For any chosen nonnegative constants μ_k not all zero, matrix $H := (\sum_k \mu_k\cdot M_k)/(\sum_k \mu_k)$ (‡) is identified with an ellipsoid $\mathbb{H}$ satisfying $\cup_k \mathbb{M}_k \supseteq \mathbb{H} \supseteq \cap_k \mathbb{M}_k$ and $\partial\mathbb{H} \supseteq \cap_k \partial\mathbb{M}_k$.

This assertion is easy to verify as follows: $x \in \cap_k \mathbb{M}_k$ if and only if every $x'\cdot M_k\cdot x \le 1$, and then $x'\cdot H\cdot x = (\sum_k \mu_k\cdot x'\cdot M_k\cdot x)/(\sum_k \mu_k) \le 1$ too, so $x \in \mathbb{H} \supseteq \cap_k \mathbb{M}_k$ as claimed. On the other hand, if $x \in \mathbb{H}$ then $(\sum_k \mu_k\cdot x'\cdot M_k\cdot x)/(\sum_k \mu_k) = x'\cdot H\cdot x \le 1$; then at least one $x'\cdot M_K\cdot x \le 1$ so that $x \in \mathbb{M}_K$, which implies that the union $\cup_k \mathbb{M}_k \supseteq \mathbb{H}$ as claimed. Finally if $\cap_k \partial\mathbb{M}_k$ is not empty it consists of all x for which every $x'\cdot M_k\cdot x = 1$, so $x'\cdot H\cdot x = (\sum_k \mu_k\cdot x'\cdot M_k\cdot x)/(\sum_k \mu_k) = 1$ too, putting $x \in \partial\mathbb{H}$ and confirming that $\partial\mathbb{H} \supseteq \cap_k \partial\mathbb{M}_k$. Thus is our formula (‡) vindicated.

The search for a small ellipsoid circumscribing $\cap_k \mathbb{M}_k$ might plausibly begin among ellipsoids $\mathbb{H}$ generated by our simple formula (‡). However, if $\cap_k \partial\mathbb{M}_k$ is empty, which seems more likely than not when the given collection has more than two ellipsoids $\mathbb{M}_k$, then that formula's smallest $\mathbb{H}$ can be bigger than the smallest circumscribing ellipsoid. Here is an example:

Example 1: Formula (‡)'s Ellipsoid can be Too Big

When row $w' \ne o'$ the matrix $W := w \cdot w'$ belongs to a degenerate ellipsoid $\mathbb{W}$ consisting of a slab between two parallel faces whose equations are $w' \cdot x = \pm 1$. In two dimensions this slab is actually a ribbon between two parallel lines. Example 1 is the intersection of three such ribbons; it is a hexagon in the plane. The three ribbons regarded as degenerate ellipses $\mathbb{M}_k$ have matrices

$$M_k := m_k \cdot m_k' \quad \text{where} \quad m_1' := [0,\ 2]\ , \quad m_2' := [\sqrt{3},\ 1]\ , \quad m_3' := [-\sqrt{3},\ 1]\ .$$

The vertices of $\mathbb{M}_1 \cap \mathbb{M}_2 \cap \mathbb{M}_3$ have coordinate columns $\pm[1/\sqrt{3},\ 0]'$, $\pm[-1/\sqrt{12},\ 1/2]'$ and $\pm[1/\sqrt{12},\ 1/2]'$. Because $\partial\mathbb{M}_1 \cap \partial\mathbb{M}_2 \cap \partial\mathbb{M}_3$ is empty, the one ellipse through all six vertices, namely the circle of radius $1/\sqrt{3}$, lies strictly inside the smallest ellipse $\mathbb{H}$ generated by our formula (‡), namely a circle of radius $1/\sqrt{2}$ whose matrix $H = (M_1 + M_2 + M_3)/3 = 2 \cdot I$.

In general, every $\mathbb{H}$ generated by formula (‡) may be too big if $\cap_k \partial\mathbb{M}_k$ is empty. How much too big can the smallest such $\mathbb{H}$ be? Other than that $\cup_k \mathbb{M}_k \supseteq \mathbb{H}$, I don't know.

Tightness

Any ellipsoid $\mathbb{H}$ circumscribing $\cap_k \mathbb{M}_k$ shall be called "Tight" just when no other ellipsoidal surface can come between $\partial\mathbb{H}$ and $\cap_k \mathbb{M}_k$. "Smallest" implies "Tight" but "Tight" need not imply "Small". For instance, the finite intersection $\mathbb{M}_1 \cap \mathbb{M}_2$ of a cylinder $\mathbb{M}_1$ of elliptical cross section with a slab $\mathbb{M}_2$ perpendicular to the cylinder's axis is circumscribed Tightly by each infinitely big $\mathbb{M}_k$. All other ellipsoids $\mathbb{H}$ generated by (‡) turn out Tight and finite.

So, some Tight ellipsoids are not small at all. Even the smallest Tight ellipsoid can extend well beyond $\cap_k \mathbb{M}_k$ if the dimension n is big enough. Here is an example:

Example 2: Tight but Not Small

Fritz John's Ellipsoid Theorem (1948) says, among other things, that an ellipsoid $\mathbb{H}$ of least *Content* (area, volume, …) circumscribing any given n-dimensional centrally symmetric bounded convex body $\mathbb{P} = -\mathbb{P}$ must satisfy $\mathbb{H} \supseteq \mathbb{P} \supseteq \mathbb{H}/\sqrt{n}$. No divisor smaller than $\sqrt{n}$ is valid here when $\mathbb{P}$ is an n-dimensional parallelepiped. Such a $\mathbb{P}$ is the intersection of n slabs $\mathbb{M}_k$ each a degenerate ellipsoid. Our formula (‡) for H generates an (n–1)-parameter family of Tightly circumscribing ellipsoids $\partial\mathbb{H}$ each of which passes through $\cap_k \partial\mathbb{M}_k$, which consists of all the vertices of $\mathbb{P}$, and yet every $\mathbb{H}$ extends beyond $\mathbb{P}$ in some directions by a factor no less than $\sqrt{n}$; one of the formula's smaller ellipsoids $\mathbb{H}$ is Fritz John's ellipsoid of least content.

Ellipsoids that Circumscribe Tightly the Intersection of Two Ellipsoids

To attenuate notational clutter let's do away with superfluous subscripts. Identify n-dimensional solid ellipsoid $\mathbb{M} := \{ x : x'\cdot M\cdot x \le 1 \}$ with positive (semi)definite matrix M, and likewise $\mathbb{W}$ with W, $\mathbb{H}$ with H and $\mathbb{T}$ with T. Our objective is to circumscribe intersection $\mathbb{M}\cap\mathbb{W}$ as closely as possible by ellipsoidal surfaces $\partial\mathbb{T}$. When " $\mathbb{T} \supseteq \mathbb{H} \supseteq \mathbb{M}\cap\mathbb{W}$ " implies " $\mathbb{H} = \mathbb{T}$ " we declare that $\mathbb{T}$ (actually $\partial\mathbb{T}$) circumscribes $\mathbb{M}\cap\mathbb{W}$ Tightly, which means so closely that no other ellipsoidal surface can slip between $\partial\mathbb{T}$ and $\mathbb{M}\cap\mathbb{W}$; and then we call $\mathbb{T}$ "Tight" too. That Tight ellipsoids exist follows from the monotonicity, boundedness, closure and therefore convergence of the matrices of any nested sequence of ellipsoids all circumscribing $\mathbb{M}\cap\mathbb{W}$.

To preclude trivialities, we assume henceforth that neither $\mathbb{M} \supseteq \mathbb{W}$ nor $\mathbb{W} \supseteq \mathbb{M}$; otherwise only the smaller of $\mathbb{M}$ and $\mathbb{W}$ could circumscribe $\mathbb{M}\cap\mathbb{W}$ Tightly. Our assumption ensures that $\partial\mathbb{M}\cap\partial\mathbb{W}$ is nonempty; it consists of two pairs of antipodal points if n = 2 and otherwise a continuum (curve, surface, …) or two, the intersection of ellipsoids $\partial\mathbb{M}$ and $\partial\mathbb{W}$ with the cone whose equation is $x'\cdot(M–W)\cdot x = 0$. Because this intersection turns out to lie within all ellipsoidal surfaces $\partial\mathbb{T}$ that circumscribe $\mathbb{M}\cap\mathbb{W}$ Tightly, they are confined narrowly thus:

Theorem

The matrix T of every Tight ellipsoid $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ and only these is generated by the formula

$$T := \lambda\cdot W + (1–\lambda)\cdot M \quad \text{as } \lambda \text{ runs from } 0 \text{ up to } 1 . \qquad (\dagger)$$

Before proving the Theorem analytically we should appreciate geometrically why it keeps λ within $0 \le \lambda \le 1$. We know already why (†), like (‡), ensures that $\partial\mathbb{T} \supseteq \partial\mathbb{M}\cap\partial\mathbb{W}$; and our anti-triviality assumption ensures that a boundary point $b \in \partial\mathbb{M}\cap\partial\mathbb{W}$ does exist. The outward normals to $\partial\mathbb{W}$ and $\partial\mathbb{M}$ at b turn out to be respectively $W\cdot b$ and $M\cdot b$. A normal to any plane supporting $\mathbb{M}\cap\mathbb{W}$ at b must be a nonnegatively weighted average of those outward normals lest the alleged support-plane actually dig inside $\mathbb{M}$ or $\mathbb{W}$ near b; draw pictures to see why. This support-plane is tangent to some Tightly circumscribing $\partial\mathbb{T}$ at b just when its outward normal $T\cdot b$ is the same nonnegatively weighted average of those outward normals $W\cdot b$ and $M\cdot b$. The Theorem's $T = \lambda\cdot W + (1–\lambda)\cdot M$, so the coefficients λ and $(1–\lambda)$ must be the nonnegatively weighted average's weights.

The foregoing paragraph's slightly circular argument does not figure in the Theorem's proof but serves merely to help explain why $0 \le \lambda \le 1$. The argument serves also to illuminate how the Theorem helps us find one of the smaller circumscribing ellipsoids when $\mathbb{M}\cap\mathbb{W}$ extends much farther in some directions than others. Among the smaller circumscribing ellipsoids are some that are Tight. Choosing one is tantamount to choosing λ. Ideal choices are determined from the outermost boundary points b of $\partial\mathbb{M}\cap\partial\mathbb{W}$ because they can be shown with the aid of Lagrange multipliers to satisfy $b/\|b\|^2 = (\lambda\cdot W + (1–\lambda)\cdot M)\cdot b$ when $\|b\|^2 := b'\cdot b$ is maximized. These equations' geometrical interpretation is that b is normal to the smallest sphere circumscribing $\mathbb{M}\cap\mathbb{W}$ and to a Tight ellipsoid $\mathbb{T}$ inside it both touching $\partial\mathbb{M}\cap\partial\mathbb{W}$ at b. However this ideal choice for λ is impractical because it requires an outermost boundary point b to be computed first, and b costs too much to compute. Later we shall investigate approximations to the ideal.

The earlier version of this work published in 1968 assumed all ellipsoids bounded since all their matrices were positive definite, thus avoiding the complications posed by infinite cylinders and slabs. A reappraisal of their utility has been brought about by experience since then, and now those complications must be addressed. Here is the first complication:

A Notational Complication

Everyone agrees that a real symmetric n-by-n matrix $M = M'$ be called "Positive Definite" just when $x'\cdot M\cdot x > 0$ for every n-vector $x \neq o$. We call M "Positive Semidefinite" just when $x'\cdot M\cdot x \geq 0$ for every x and $z'\cdot M\cdot z = 0$ for some $z \neq o$; many other users of the term "Positive Semidefinite" omit the requirement that any such z exist. To accommodate their ambiguity we call M "Positive (Semi)Definite" just when $x'\cdot M\cdot x \geq 0$ for every x no matter whether z exists.

Nullspace Revelation

If matrix $M = M'$ is positive (semi)definite, and if $z'\cdot M\cdot z = 0$, then $M\cdot z = o$.

Proof: Because $0 \leq (z - ß\cdot M\cdot z)'\cdot M\cdot(z - ß\cdot M\cdot z)/ß = -2\cdot(M\cdot z)'\cdot(M\cdot z) + ß\cdot(M\cdot z)'\cdot M\cdot(M\cdot z)$ for every $ß > 0$, we find $0 \leq (M\cdot z)'\cdot(M\cdot z) \leq ß\cdot(M\cdot z)'\cdot M\cdot(M\cdot z)/2 \to 0+$ as $ß \to 0+$, so $M\cdot z = o$.

Nullspace revelation will figure in the removal of the next complication, which is the possibility that $\mathbb{M}\cap\mathbb{W}$ extends to infinity; it will be removed after the Corollary below.

The Theorem's proof relies solely upon the connection between the geometry of ellipsoids $\mathbb{T}$, $\mathbb{M}$, $\mathbb{W}$, $\mathbb{H}$, … and the algebra of their respective n-by-n positive (semi)definite matrices T, M, W, H, … . Summarized succinctly, the connection goes thus:

Lemma

$\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ if and only if $x'\cdot T\cdot x \leq \max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}$ for all column n-vectors x.

Proof: If $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ then whenever $ß := \max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\} \neq 0$ we find $x/\sqrt{ß} \in \mathbb{M}\cap\mathbb{W}$ whence $x/\sqrt{ß} \in \mathbb{T}$ and therefore $x'\cdot T\cdot x \leq ß$; however when $ß = 0$ then, for all $\mu \neq 0$, we find in turn that $(x/\mu)'\cdot M\cdot(x/\mu) = (x/\mu)'\cdot W\cdot(x/\mu) = 0$, $x/\mu \in \mathbb{M}\cap\mathbb{W}$, $x/\mu \in \mathbb{T}$, $x'\cdot T\cdot x \leq \mu^2$ and finally $x'\cdot T\cdot x = 0 = ß$ after $\mu \to 0$. Conversely, if $x'\cdot T\cdot x \leq ß := \max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}$ for all x, and if $x \in \mathbb{M}\cap\mathbb{W}$, then $x'\cdot T\cdot x \leq ß \leq 1$ so $x \in \mathbb{T}$ and therefore $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$.

Corollary

$\mathbb{T} \supseteq \mathbb{M}$ if and only if $x'\cdot T\cdot x \leq x'\cdot M\cdot x$ for all column n-vectors x.

Proof: Regard the whole vector-space as an ellipsoid $\mathbb{W}$ whose matrix W is the zero matrix O.

Now we can remove the complicating possibility that $\mathbb{M}\cap\mathbb{W}$ extends to infinity, which happens only when some $z \neq o$ satisfies $z'\cdot M\cdot z = z'\cdot W\cdot z = 0$. When this happens it reveals the existence of a proper subspace **Z** consisting of all vectors z that satisfy $M\cdot z = W\cdot z = o$; this **Z** is the nonzero intersection of the nullspaces of M and of W. The Lemma implies for any $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ that $z'\cdot T\cdot z = 0$ too and reveals that **Z** is contained in the nullspace of T. Embed a basis for **Z** in any new basis for the whole space and change to new coordinates using this new basis. Doing so transforms M, W and T into new *Congruent* matrices $\begin{bmatrix} O & O \\ O & \mathrm{M} \end{bmatrix}$, $\begin{bmatrix} O & O \\ O & \mathrm{W} \end{bmatrix}$ and $\begin{bmatrix} O & O \\ O & \mathrm{T} \end{bmatrix}$ respectively in which the new smaller versions M, W and T have the same properties as had the old larger versions except that now $x'\cdot M\cdot x$ and $x'\cdot W\cdot x$ cannot both vanish at the same vector $x \neq o$. The old anti-triviality assumption, namely that $x'\cdot M\cdot x > x'\cdot W\cdot x$ for some x but $x'\cdot M\cdot x < x'\cdot W\cdot x$ for others, also persists for the new smaller matrices. Geometrically, replacing old by new amounts to a projection parallel to **Z** of the original vector-space onto any subspace complementary to **Z**, thus conveying all containment relations from parallel cylinders and slabs in the full space to their respective intersections with that complementary subspace, wherein we remain henceforth.

In short, we now enjoy two simplifying assumptions:

- The nullspaces of the given positive (semi)definite matrices M and W intersect only in o, implying that $\mathbb{M}\cap\mathbb{W}$ is finite and $\lambda\cdot W + (1-\lambda)\cdot M$ is positive definite for $0 < \lambda < 1$.
- $x'\cdot M\cdot x > x'\cdot W\cdot x$ for some x but $x'\cdot M\cdot x < x'\cdot W\cdot x$ for others, so $\partial\mathbb{M}\cap\partial\mathbb{W}$ is nonempty, thus precluding trivial cases like dimension $n = 1$ and others that imply $\lambda = \lambda^2$ in (†) because the only Tight ellipsoid is whichever of $\mathbb{M}$ and $\mathbb{W}$ is included in the other.

These assumptions will simplify the Theorem's proof without detracting from its utility though the proof remains complicated by its dependence upon a complicated procedure:

Interposition Procedure

Given the positive (semi)definite n-by-n matrix H of any ellipsoid $\mathbb{H} \supseteq \mathbb{M}\cap\mathbb{W}$, Tight or not, the procedure laid out hereunder determines λ within $0 \le \lambda \le 1$ to *interpose* the ellipsoidal surface $\partial\mathbb{T}$ belonging to $T := \lambda\cdot W + (1-\lambda)\cdot M$ between $\mathbb{H}$ and $\mathbb{M}\cap\mathbb{W}$, so $\mathbb{H} \supseteq \mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$. This T, the Theorem's T in (†), will turn out to be Tight though not during the procedure, which will merely be proved feasible albeit impractical. Note that the anti-triviality assumption implies $n \ge 2$, which makes the procedure's first step computationally nontrivial:

$$\text{Determine} \quad Ç := \inf_{x\ne o} \max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}/x'\cdot H\cdot x\ .$$

$Ç \ge 1$ because the Lemma applies to $\mathbb{H} \supseteq \mathbb{M}\cap\mathbb{W}$. Although the search for Ç has to avoid any $x \ne o$ at which $H\cdot x = o$, infimum Ç is actually an attained minimum. It is so because M+W is positive definite and $\max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}/x'\cdot H\cdot x$ is homogeneous of degree 0 in x, whence follows

$$\begin{aligned} Ç &= \inf\big(\max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}/x'\cdot H\cdot x\big) && \text{sought over } (M+W)\cdot x \ne o \\ &= \inf\big(\max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}/x'\cdot H\cdot x\big) && \text{sought over } x'\cdot(M+W)\cdot x = 1 \\ &\ge \inf\big((1/2)/x'\cdot H\cdot x\big) && \text{sought over } x'\cdot(M+W)\cdot x = 1\ . \end{aligned}$$

This means that the search for Ç can be confined to a closed bounded region on the ellipsoidal surface whereon $x'\cdot(M+W)\cdot x = 1$ from which has been excised the open (perhaps empty) region wherein $x'\cdot H\cdot x < 1/(2Ç)$. Therefore Ç is the attained minimum of a continuous function on a compact set and is attained thereon at some vector $x = c$ where $H\cdot c \ne o$. After M and W have been swapped if necessary, this c will satisfy $c'\cdot H\cdot c > 0$ and, with $Ç \ge 1$, also

$$c'\cdot W\cdot c/c'\cdot H\cdot c \le Ç = c'\cdot M\cdot c/c'\cdot H\cdot c \le \max\{x'\cdot M\cdot x,\ x'\cdot W\cdot x\}/x'\cdot H\cdot x \ \text{ for all } x \ne 0\ . \qquad (\#)$$

Now the procedure splits into three cases according to whether $c'\cdot W\cdot c = c'\cdot M\cdot c$ and then, if so, whether $W\cdot c = M\cdot c$. In each case the procedure will determine λ to satisfy

$$0 \le \lambda \le 1 \ \text{ and } \ T := \lambda\cdot W + (1-\lambda)\cdot M \ \text{ and } \ x'\cdot T\cdot x \ge Ç\cdot x'\cdot H\cdot x \ge x'\cdot H\cdot x \ \text{ for all } x\ . \qquad (\lozenge)$$

• **Case 1:** Suppose $c'\cdot W\cdot c < c'\cdot M\cdot c = \text{Ç}\cdot c'\cdot H\cdot c$ in (#) above. In this case $T := M$ for $\lambda := 0$ will be seen to satisfy (◊) as follows:

Given any vector x and scalar $\text{ß} \neq 0$ let $y := c - \text{ß}\cdot x$, and then invoke (#) to infer that

$$y'\cdot W\cdot y < y'\cdot M\cdot y \quad \text{and so} \quad \text{Ç}\cdot y'\cdot H\cdot y \le y'\cdot M\cdot y \quad \text{for all sufficiently small } |\text{ß}| > 0 .$$

Choose the sign of such a ß to make $\text{ß}\cdot x'\cdot(M - \text{Ç}\cdot H)\cdot c \ge 0$ too. Then it will make

$$0 \le y'\cdot(M - \text{Ç}\cdot H)\cdot y = \text{ß}^2\cdot x'\cdot(M - \text{Ç}\cdot H)\cdot x - 2\text{ß}\cdot x'\cdot(M - \text{Ç}\cdot H)\cdot c \le \text{ß}^2\cdot x'\cdot(M - \text{Ç}\cdot H)\cdot x .$$

This implies $x'\cdot T\cdot x = x'\cdot M\cdot x \ge \text{Ç}\cdot x'\cdot H\cdot x$ for every x, so $T = M$ satisfies (◊).

• **Cases 2 and 3:** Suppose $c'\cdot W\cdot c = c'\cdot M\cdot c = \text{Ç}\cdot c'\cdot H\cdot c$ in (#) above. Given any vector x and scalar ß let $y := x + \text{ß}\cdot c$ and invoke (#) to infer from $\text{Ç}\cdot y'\cdot H\cdot y \le \max\{y'\cdot M\cdot y,\ y'\cdot W\cdot y\}$ that

$$\text{Ç}\cdot x'\cdot H\cdot x + 2\text{ß}\cdot\text{Ç}\cdot x'\cdot H\cdot c \le \max\{x'\cdot M\cdot x + 2\text{ß}\cdot x'\cdot M\cdot c,\ x'\cdot W\cdot x + 2\text{ß}\cdot x'\cdot W\cdot c\} \quad \text{for all } x \text{ and ß}. \qquad (*)$$

One implication of this inequality (*) for cases 2 and 3 is that $\text{Ç}\cdot H\cdot c = \lambda\cdot W\cdot c + (1-\lambda)\cdot M\cdot c$ for some scalar λ; it comes about as follows: Consider a vector $r := \lambda\cdot W\cdot c + \omega\cdot M\cdot c - \text{Ç}\cdot H\cdot c$ for scalars λ and ω chosen to satisfy $r'\cdot W\cdot c = r'\cdot M\cdot c = 0$. Such scalars exist because they satisfy the *Normal Equations* for a *Least-Squares Problem*

“ Choose λ and ω to minimize $r'\cdot r = (\lambda\cdot W\cdot c + \omega\cdot M\cdot c - \text{Ç}\cdot H\cdot c)'\cdot(\lambda\cdot W\cdot c + \omega\cdot M\cdot c - \text{Ç}\cdot H\cdot c)$ ”

that can always be solved for finite values λ and ω though not necessarily uniquely. When r is substituted for x in (*) it satisfies $\text{Ç}\cdot r'\cdot H\cdot r + 2\text{ß}\cdot r'\cdot r \le \max\{r'\cdot M\cdot r,\ r'\cdot W\cdot r\}$ for all ß. Letting $\text{ß} \to +\infty$ reveals that $r = o$; in other words $\text{Ç}\cdot H\cdot c = \lambda\cdot W\cdot c + \omega\cdot M\cdot c$. Premultiplying this equation by c' to get $1 = \lambda + \omega$ confirms the assertion above that $\text{Ç}\cdot H\cdot c = \lambda\cdot W\cdot c + (1-\lambda)\cdot M\cdot c$ for some scalar λ in cases 2 and 3. We have not yet proved $0 \le \lambda \le 1$.

• **Case 2:** Suppose $c'\cdot W\cdot c = c'\cdot M\cdot c = \text{Ç}\cdot c'\cdot H\cdot c$ in (#) above but $W\cdot c \neq M\cdot c$. In this case $W\cdot c$ and $M\cdot c$ are linearly independent because otherwise, were $W\cdot c = \omega\cdot M\cdot c$ for some scalar ω, say, premultiplying by c' and invoking this case’s suppositions would produce a contradictory $\omega = 1$. This linear independence will constrain λ in the equation $\text{Ç}\cdot H\cdot c = \lambda\cdot W\cdot c + (1-\lambda)\cdot M\cdot c$ to satisfy $0 \le \lambda \le 1$ as follows: Choose any vector v for which $v'\cdot W\cdot c > 0 > v'\cdot M\cdot c$; one such choice is $v := W\cdot c/\sqrt{(c'\cdot W^2\cdot c)} - M\cdot c/\sqrt{(c'\cdot M^2\cdot c)}$. Substitute v for x in (*), replace $\text{Ç}\cdot H\cdot c$ there by $\lambda\cdot W\cdot c + (1-\lambda)\cdot M\cdot c$, and let ß approach first $-\infty$ and then $+\infty$ to deduce first that $v'\cdot M\cdot c \le \lambda\cdot v'\cdot W\cdot c + (1-\lambda)\cdot v'\cdot M\cdot c \le v'\cdot W\cdot c$ and then that $0 \le \lambda \le 1$ as has just been claimed.

Now set $T := \lambda\cdot W + (1-\lambda)\cdot M$. Its ellipsoid $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ because of (‡). Next we shall see why $x'\cdot T\cdot x \ge \text{Ç}\cdot x'\cdot H\cdot x \ge x'\cdot H\cdot x$ for all vectors x, but beginning with almost all.

Given any x for which $c'\cdot(W-M)\cdot x \neq 0$ set $\text{ß} := x'\cdot(W-M)\cdot x/c'\cdot(W-M)\cdot x$ and $y := x - \text{ß}\cdot c/2$. After $y'\cdot W\cdot y = y'\cdot M\cdot y = y'\cdot T\cdot y$ has been confirmed, substituting y for x in (#) implies $y'\cdot T\cdot y \ge \text{Ç}\cdot y'\cdot H\cdot y$, which this case’s suppositions about c and $\text{Ç}\cdot H\cdot c = T\cdot c$ transform into the desired inequality $x'\cdot T\cdot x \ge \text{Ç}\cdot x'\cdot H\cdot x$ now valid for all x except maybe those in the plane whose equation is $c'\cdot(W-M)\cdot x = 0$. Continuity eliminates this exception, so (◊) is true this case.

• **Case 3:** Suppose $c'\cdot W\cdot c = c'\cdot M\cdot c = \text{Ç}\cdot c'\cdot H\cdot c$ in (#) above and $W\cdot c = M\cdot c$. This case will invoke the Interposition Procedure recursively upon the (n–1)-dimensional subspace **¥** of all vectors y satisfying $c'\cdot H\cdot y = 0$ ($= c'\cdot M\cdot y = c'\cdot W\cdot y$ since $\text{Ç}\cdot H\cdot c = W\cdot c = M\cdot c$ now)

Every vector x in the whole space possesses a unique decomposition $x = \text{ß}\cdot c + y$ with $y \in \text{¥}$; and both $x'\cdot(W - \text{Ç}\cdot H)x = y'\cdot(W - \text{Ç}\cdot H)y$ and $x'\cdot(M - \text{Ç}\cdot H)x = y'\cdot(M - \text{Ç}\cdot H)y$ because of this case's suppositions. Consequently the decomposition projects inequalities like (#) and (◊) from all x in the whole space onto analogous inequalities satisfied by all y in the subspace ¥, thus reducing the task of finding $T := \lambda\cdot W + (1-\lambda)\cdot M$ with $0 \le \lambda \le 1$ to the same task but satisfying $\text{ç}\cdot y'\cdot H\cdot y \le y'\cdot T\cdot y$ only for all $y \in \text{¥}$ given $\text{ç} := \inf_{o \ne y \in \text{¥}} \max\{y'\cdot M\cdot y,\ y'\cdot W\cdot y\}/y'\cdot H\cdot y \ge \text{Ç}$. To interpret this in terms of matrices rather than subspaces, choose any basis for ¥ and append c to it to get a new basis for the whole space. Then change to new coordinates using this new basis to transform M, W, H and T into congruent matrices respectively $\begin{bmatrix} \mathrm{M} & o \\ o' & \omega \end{bmatrix}$, $\begin{bmatrix} \mathrm{W} & o \\ o' & \omega \end{bmatrix}$, $\begin{bmatrix} \mathrm{H} & o \\ o' & \omega/\text{Ç} \end{bmatrix}$ and $\begin{bmatrix} \mathrm{T} & o \\ o' & \omega \end{bmatrix}$ in which $\omega = c'\cdot M\cdot c = c'\cdot W\cdot c = \text{Ç}\cdot c'\cdot H\cdot c = c'\cdot T\cdot c$ regardless of μ, and the new smaller matrices M, W, H and T play the same rôles as the old matrices did though with a new $\text{ç} \ge \text{Ç}$.

Interposition is accomplished trivially if ¥ is 1-dimensional; and otherwise the procedure is accomplished by repeating in ¥ (upon the new smaller matrices) the calculations carried out above for the whole space (upon the original matrices). Thus ends the Interposition Procedure.

Proof of the Theorem

On the one hand, suppose H is the matrix of a Tight ellipsoid $\mathbb{H} \supseteq \mathbb{M}\cap\mathbb{W}$. The foregoing interposition procedure chooses λ in $0 \le \lambda \le 1$ to produce the matrix $T := \lambda\cdot W + (1-\lambda)\cdot M$ of an ellipsoid T satisfying $\mathbb{H} \supseteq \mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ which, since $\mathbb{H}$ is Tight, implies that $H = T$ as the Theorem claims.

On the other hand suppose now that $\mathbb{H}$ is the ellipsoid belonging to a matrix $H = \text{ß}\cdot W + (1-\text{ß})\cdot M$ for some ß in $0 \le \text{ß} \le 1$; why must $\mathbb{H}$ be Tight as the Theorem claims? If any ellipsoid's surface $\partial\mathbb{Y}$ can slip between $\mathbb{H}$ and $\mathbb{M}\cap\mathbb{W}$, the interposition procedure chooses again λ in $0 \le \lambda \le 1$ to produce another matrix $T := \lambda\cdot W + (1-\lambda)\cdot M$ of an ellipsoid $\mathbb{T}$ now satisfying $\mathbb{H} \supseteq \mathbb{Y} \supseteq \mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ which, says the Corollary, implies that $x'\cdot H\cdot x \le x'\cdot T\cdot x$ for all x. This inequality simplifies to $(\text{ß}-\lambda)\cdot x'\cdot(W-M)\cdot x \le 0$ for all x; since the anti-triviality assumption makes $x'\cdot(W-M)\cdot x$ positive for some vectors x, negative for others, the inequality forces $\lambda = \text{ß}$ and then $T = H$, and this pinches $\mathbb{H} \supseteq \mathbb{Y} \supseteq \mathbb{T} = \mathbb{H}$ to force $\mathbb{Y} = \mathbb{H}$. End of proof.

Including the Interposition Procedure, the proof takes over three pages. Must it be so long?

The Smallest Ellipsoids Circumscribing the Intersection of Two Ellipsoids

So long as "smaller than" implies "contained within", the smallest circumscribing ellipsoids must be found among the Tight ellipsoids, and the Theorem exhibits all of these. Which of these is smallest depends upon what "smallest" means. Three possibilities come to mind. The first, "smallest in content", is independent of the choice of basis because the ratio of one body's content to another's does not change when coordinates change from one basis to another. The other possibilities make sense in Euclidean spaces equipped, as every Euclidean space can be, with an orthonormal basis. A change of basis will help us examine all three possibilities.

Any two real symmetric positive (semi)definite n-by-n matrices W and M can be diagonalized simultaneously by any one of infinitely many congruences; this is summarized succintly in §8.7 of the text by Golub and Van Loan (1996). Each congruence is a change of basis that transforms W and M into $C'\cdot W\cdot C = \mathrm{Diag}[\omega_j]$ and $C'\cdot M\cdot C = \mathrm{Diag}[\mu_j]$ for one of infinitely many suitable invertible n-by-n matrices C. No matter which of these is chosen, the same (multi)set $\{\mu_j/\omega_j\}$ of n ratios, not necessarily all distinct nor all finite, will be obtained. The same congruence also diagonalizes the Tight ellipsoids' matrices $T := \lambda\cdot W + (1-\lambda)\cdot M$, transforming them into $C'\cdot T\cdot C = \mathrm{Diag}[\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j]$. So far as choosing λ to minimize the content of a Tight ellipsoid $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ is concerned, the given matrices W and M might as well have been given diagonalized already, letting $C := I$.

Minimizing a Tight Ellipsoid's Content

So far as choosing λ to minimize the content of a Tight ellipsoid $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$ is concerned, those coordinate directions for which $\omega_j = \mu_j$ might as well be disregarded since corresponding entries in the diagonal $C'\cdot T\cdot C$ do not change when λ changes. Deleting these equal entries from all diagonals is geometrically tantamount to projecting all ellipsoids under consideration onto the lower-dimensional subspace where the shapes of Tight ellipsoids are influenced by λ. In the next paragraph we assume for each j that either $\omega_j > \mu_j \ge 0$ or $\mu_j > \omega_j \ge 0$; and each of these orderings must occur at least once lest the anti-triviality assumption be violated.

The choice λ that minimizes the content of the Theorem's Tight $\mathbb{T}$ must maximize det(T) or, equivalently, maximize $\prod_j(\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j)$. The det-maximizing λ is a zero of the derivative $f(\lambda) := d\log(\det(T))/d\lambda = \sum_j 1/(\lambda + \mu_j/(\omega_j-\mu_j))$ provided it has a zero in the interval $0 \le \lambda \le 1$. If it does, it has just one zero λ because $f(\lambda)$ is a monotone decreasing function in that interval. Otherwise the det-maximizing $\lambda := 0$ if $f(0) < 0$ or $\lambda := 1$ if $f(1) > 0$; in such cases the Tight ellipsoid $\mathbb{T}$ of smallest content is either $\mathbb{M}$ or $\mathbb{W}$. This is not unusual; an example has $n = 2$, $\mu_1/\omega_1 = 5/4$, $\mu_2/\omega_2 = 1/2$, $\lambda = 1$ and the Tight $\mathbb{T}$ of minimum area is $\mathbb{T} = \mathbb{W}$. In general, though, $\lambda < 1$ and $\mathbb{T} \ne \mathbb{W}$ if any $\omega_j = 0 < \mu_j$, and $\lambda > 0$ and $\mathbb{T} \ne \mathbb{M}$ if any $\mu_j = 0 < \omega_j$.

Example 3: Tight with Minimum Content can be Too Long

How much bigger than $\mathbb{M}\cap\mathbb{W}$ can the Tight $\mathbb{T}$ of minimum content be? Fritz John's Ellipsoid Theorem says $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W} \supseteq \mathbb{T}/\sqrt{n}$; this example shows why no divisor smaller than $\sqrt{n}$ can be valid in general: First choose any tiny positive ε, the tinier the better, and then set every $\omega_j := \varepsilon$ and every $\mu_j := 1$ except $\mu_n := 0$; now $f(\lambda) = (n-1)/(\lambda - 1/(1-\varepsilon)) + 1/\lambda$ vanishes at $\lambda := 1/(n\cdot(1-\varepsilon))$, making $T := \lambda\cdot W + (1-\lambda)\cdot M = \mathrm{Diag}[1-1/n, 1-1/n, \dots, 1-1/n, \varepsilon/(n\cdot(1-\varepsilon))]$. Described geometrically, this example has an infinite circular cylinder $\mathbb{M}$ of radius 1, a huge sphere $\mathbb{W}$ of radius $1/\sqrt{\varepsilon}$, a long rod $\mathbb{M}\cap\mathbb{W}$ cut from the cylinder by the sphere, and a longer cigar-shaped ellipsoid $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W} \supseteq \mathbb{T}/\sqrt{n\cdot(1-\varepsilon)}$. The width of $\mathbb{T}$ exceeds the width of the cylindrical rod $\mathbb{M}\cap\mathbb{W}$ by a modest factor $1/\sqrt{1-1/n}$, but the length of $\mathbb{T}$ exceeds the rod's length by a large factor $\sqrt{n\cdot(1-\varepsilon)}$ when the space's dimension n is big.

The circumscribing ellipsoid of smallest content need not be nearly smallest in any other sense.

Minimizing a Tight Ellipsoid's Box-Diameter
Henceforth $\mathbb{M}\cap\mathbb{W}$ is assumed bounded for the sake of a slightly simpler exposition.

In an n-dimensional Euclidean space the first gauge that comes to mind to measure the size of an ellipsoid $\mathbb{T}$ is the length of its major axis, which turns out to be $2/\sqrt{(\text{minimum eigenvalue of } T)}$ where T is the positive definite matrix belonging to $\mathbb{T} := \{x:\ x'\cdot T\cdot x \le 1\}$. Of all Tight $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$, the smallest by this gauge is found by choosing λ in $0 \le \lambda \le 1$ to maximize the least eigenvalue of $T := \lambda\cdot W + (1-\lambda)\cdot M$. Like so many other ideas that come first to mind, this gauge turns out to be a poor idea. Besides being costly to compute, the Tight $\mathbb{T}$ with the smallest major axis tends to rotundity wherever $\mathbb{M}\cap\mathbb{W}$ is long and narrow. For instance, in Example 3 the major axis of the Tight $\mathbb{T}$ is minimized when $\lambda := 1$ and then $\mathbb{T} = \mathbb{W}$ is the huge sphere of diameter $2/\sqrt{\varepsilon}$ far fatter than the slender rod $\mathbb{M}\cap\mathbb{W}$ whose thickness is 2.

A gauge better for our purposes should strike a compromise between the excessive width of the Tight $\mathbb{T}$ with the smallest major axis, and the excessive length of the Tight $\mathbb{T}$ with the least content. And an affordable computational cost is another attribute we desire for the $\mathbb{T}$ smallest by a better gauge. Here is a candidate:

Let's abbreviate "rectangular parallelepiped" to "box". A box's diameter is the length of any of its interior diagonals. Define the *Box-Diameter* $Bd(\mathbb{T})$ of a body $\mathbb{T}$ to be the least of the diameters of its circumscribing boxes. To compute the Box-Diameter $Bd(\mathbb{T})$ of an ellipsoid $\mathbb{T}$ from its matrix T turns out to be comparatively easy: $Bd(\mathbb{T}) := 2\sqrt{(\text{Trace}(T^{-1}))}$. Moreover every box that barely circumscribes ellipsoid $\mathbb{T}$, touching it with every face, turns out to have that same least diameter. In Euclidean space $Bd(\mathbb{T})$ seems to sum up concisely the overall size of an ellipsoid $\mathbb{T}$, and its computational cost is tolerable.

To minimize $Bd(\mathbb{T})$ among Tight $\mathbb{T} \supseteq \mathbb{M}\cap\mathbb{W}$, we must choose λ for $T := \lambda\cdot W + (1-\lambda)\cdot M$ to minimize $\text{Trace}(T^{-1})$. The congruence that transformed T into $C'\cdot T\cdot C = \text{Diag}[\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j]$ above produces

$$\text{Trace}(T^{-1}) = \text{Trace}(C'^{-1}\cdot\text{Diag}[\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j]^{-1}\cdot C^{-1})$$
$$= \text{Trace}((C'\cdot C)^{-1}\cdot\text{Diag}[\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j]^{-1}) = \textstyle\sum_j \theta_j/(\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j)$$

wherein θ_j is the j^{th} diagonal element of $(C'\cdot C)^{-1}$. Every $\theta_j > 0$. The minimizing λ is a zero of the derivative $\Theta(\lambda) := d\,\text{Trace}(T^{-1})/d\lambda = \sum_j \theta_j\cdot(\mu_j-\omega_j)/(\lambda\cdot\omega_j + (1-\lambda)\cdot\mu_j)^2$ provided it has a zero in the interval $0 \le \lambda \le 1$. If it does, it has just one zero λ because $\Theta(\lambda)$ is a monotone increasing function in that interval. Otherwise the minimizing $\lambda := 0$ if $\Theta(0) > 0$ or $\lambda := 1$ if $\Theta(1) < 0$; in such cases the Tight ellipsoid $\mathbb{T}$ of smallest box-diameter is either $\mathbb{M}$ or $\mathbb{W}$. This is not unusual; an example has $n = 2$, $\theta_1 = \theta_2 = 1$, $\mu_1/\omega_1 = 5/4$, $\mu_2/\omega_2 = 1/2$, $\lambda = 1$ and the Tight $\mathbb{T}$ of minimum box-diameter is $\mathbb{T} = \mathbb{W}$. In general, though, $\lambda < 1$ and $\mathbb{T} \ne \mathbb{W}$ if any $\omega_j = 0 < \mu_j$, and $\lambda > 0$ and $\mathbb{T} \ne \mathbb{M}$ if any $\mu_j = 0 < \omega_j$.

In the excluded case, when some $\omega_j = \mu_j = 0$, the foregoing computations must be preceded by the orthogonal projection of the vector space upon the orthogonal complement of the intersection **Z** of the nullspaces of M and of W; compare the paragraph after the Corollary above.

Example 3 Revisited: Tight, Thin and Not Too Long

This example had n-by-n matrices $C := I$, $W := \varepsilon\cdot I$ and $M := I - u\cdot u'$ for any very tiny $\varepsilon > 0$ and $u' := [0, 0, 0, \dots, 0, 1]$. Now $\Theta(\lambda) = (n-1)\cdot(1-\varepsilon)/\big(1 - (1-\varepsilon)\cdot\lambda\big)^2 - 1/(\varepsilon\cdot\lambda^2)$ vanishes at $\lambda := \big(1 - \sqrt{(n-1)\cdot\varepsilon/(1-\varepsilon)}\big)/(1 - n\cdot\varepsilon)$ provided $0 < \varepsilon < 1/n$. This λ minimizes $\mathrm{Bd}(\mathbb{T})$ for $T := \lambda\cdot W + (1-\lambda)\cdot M$. To simplify the comparison of this ellipsoid $\mathbb{T}$ with the long narrow rod $\mathbb{M}\cap\mathbb{W}$, suppose that its dimension n is huge and ε is still negligible compared with $1/n$.

Then the width of $\mathbb{T}$ is about $2/\sqrt{(n-1)\cdot\varepsilon}$ which is much bigger than the width 2 of $\mathbb{M}\cap\mathbb{W}$ but small compared with its length $2/\sqrt{\varepsilon}$, which is very slightly less than the length of $\mathbb{T}$. This Tight ellipsoid of minimized box-diameter approximates the shape of rod $\mathbb{M}\cap\mathbb{W}$ far better than did the Tight ellipsoids of either minimum diameter or minimum content, and yet this $\mathbb{T}$ does not deserve to be called "optimal". A slightly smaller λ can produce another Tight $\mathbb{T}$ at most a few percent longer but at least an order of magnitude slimmer.

Perhaps more than anything else, what this example teaches is that in ostensibly uncomplicated situations the use of simple-minded criteria for optimality can produce results far from optimal in a broader sense. But you probably know that already. Too many others don't, alas.

Remarks and References

Simultaneous diagonalization of two positive (semi)definite matrices is an eigenvalue calculation treated succinctly in §8.7 of the text *Matrix Computations* by G.H. Golub and C.F. Van Loan (1996, 3rd ed., Johns Hopkins Press, Baltimore). They supply a copious reading list too.

Fritz John's Ellipsoid Theorem (1948) and its proof covering arbitrary compact convex bodies were his contribution to the *1948 Courant Anniversary Volume* (InterScience/Wiley, New York). A direct proof for centrally symmetric convex bodies is on my web page near the end of `<http://www.cs.berkeley.edu/~wkahan/MathH110/NORMlite.pdf>`. More applications of Fritz John's Ellipsoid Theorem and another longer proof for its centrally symmetric case appear in Keith Ball's lecture notes "An Elementary Introduction to Modern Convex Geometry", pp. 1-58 of *Flavors of Geometry*, MSRI Publications - Vol. 31, edited by Silvio Levy for Cambridge University Press, Cambridge, 1997. Don't rely too much upon the title's word "Elementary". Ball's notes are also posted at `<http://www.msri.org/publications/books/Book31/files/ball.pdf>`.

In the late 1960s the application of ellipsoids to circumscribe other bodies weighed on many minds. The earlier version of this work published in 1968 cited …

- D.K. Faddeèv and V.N. Faddeèva (1968) "Stability in Linear Algebra Problems" *Proc. IFIP Congress 68* in Edinburgh.
- F.C. Schweppe (1967) "Recursive state estimation when observation errors and system inputs are bounded" Sperry Rand Research Centre Report RR-67-25, Sudbury, Mass.
- W. Kahan (1967) "Circumscribing an ellipsoid about the Minkowski sum of given ellipsoids" (Submitted to J. Linear Algebra). *I cannot recall what happened to this submission. An updated and much expanded version will be posted on my web page at* `<…/MinkoSum.pdf>`.
- W. Kahan (1968) "An ellipsoidal error bound for linear systems of differential equations" (Manuscript to appear). *Now see my web page's* `<…/Math128/Ellipsoi.pdf>`.

Parts of the material in the last two citations were quoted in the late Fred C. Schweppe's textbook *Uncertain Dynamic Systems* (1973, Prentice-Hall, NJ) though it is devoted mostly to a treatment of probabilistic uncertainty. In the late 1960s ellipsoidal bounds for errors and/or uncertainty weighed upon many minds but were considered extravagant; they augmented an n-dimensional computation of desired results with an n^2-dimensional computation of bounds for those results' uncertainties. Computation costs so little nowadays that almost any extravagance is affordable, though few practitioners are inclined to perform enormously more computation to assess a result's uncertainty than was performed merely to obtain that result.

An independent redevelopment of ellipsoidal bounds has been published by Arnold Neumaier in "The wrapping effect, ellipsoid arithmetic, stability and confidence regions" pp. 175-190 in *Computing Supplementum* **9** (1993), `<http://solon.cma.univie.ac.at/papers.html#ell>`. Further recent work along similar lines has been published by Pravin P. Varaiya and Alex A. Kurzhanskiy (*fils*) in "Ellipsoidal Techniques for Reachability Analysis of Discrete-Time Linear Systems" *IEEE Trans. Automatic Control*, to appear in 2006. Another work is "On Ellipsoidal Techniques for Reachability Analysis" by Varaiya and Alex B. Kurzhanski (*père*) in *Optimization Methods and Software* **17** (2002) pp. 177-237. They concentrate upon ascertaining the boundary of a region rather than merely circumscribing it. Much of their work is posted on Varaiya's web page: `<http://paleale.eecs.berkeley.edu/~varaiya/hybrid.html>`. Beware: They do not define "Tightly" so tightly as it is defined here, where ellipsoid $\mathbb{T} \supseteq \mathbb{B}$ Tightly just when *no other ellipsoid* $\mathbb{H}$ *whatever* can satisfy $\mathbb{T} \supseteq \mathbb{H} \supseteq \mathbb{B}$. They let $\mathbb{T}$ be called "tight" if both it and $\mathbb{H}$ are restricted to the same family of ellipsoids generated by one of their parameterized formulas.

So far as I know, a few questions that required further study in 1968 remain unanswered. What characterizes the ellipsoids that circumscribe Tightly the intersection of several ellipsoids? What characterizes ellipsoids that circumscribe Tightly the intersection of two ellipsoids with different centers? To decide numerically whether two ellipsoids, one described by $(x-w)'\cdot W\cdot(x-w) \le 1$ and the other by $(x-m)'\cdot M\cdot(x-m) \le 1$, intersect nontrivially for given numerical data w, W, m and M seems best accomplished by an eigenvalue computation: A simultaneous diagonalization of W and M is followed by a search for all the real zeros λ of a rational function somewhat like $\Theta(\lambda)$ and subsequent tests performed upon them. Cited in 1968 were …

- J.W. Burrows (1966) "Maximization of a second-degree polynomial on the unit sphere" pp. 441-4 in *Math. of Comp*. **20**.
- G.E. Forsythe and G.H. Golub (1965) "On the stationary values of a second-degree polynomial on the unit sphere" pp. 1050-1068 in *J. Soc. Indust. Appl. Math*. **13**.

In principle the same decision about intersection could be rendered by an exact computation (no rounding errors) of a few polynomial discriminants involving the data w, W, m and M; but the computational cost of those polynomials appears too horrible to contemplate so far as I know.

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